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# On the fundamental formulas of complex Finsler submanifolds☆

Chunping Zhong

*School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China*

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Dedicated to Professor Tongde Zhong on the occasion of his 80th birthday

**Abstract**

Let $M$ be a complex manifold endowed with a strongly pseudoconvex Finsler metric $F$, and $\mathcal{M}$ be a complex submanifold of $M$ endowed with the induced complex Finsler metric $\mathcal{F}$. In this paper, the Gauss, Codazzi and Ricci equations are obtained with respect to the Chern–Finsler connection on $(M, F)$, the relationship between the torsion of the induced Chern–Finsler connection and the torsion of the Chern–Finsler connection on the ambient Finsler manifold are obtained. As applications of the fundamental formulas, we first prove that the holomorphic curvature of the induced complex Finsler metric $\mathcal{F}$ does not exceed the holomorphic curvature of $F$, and then give a characterization of the totally geodesic complex Finsler submanifold in terms of the horizontal components of the second fundamental form of $(\mathcal{M}, \mathcal{F})$.


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## 1. Basic notations

Let $M$ be an $n$-dimensional complex manifold, $(z^\alpha)$ be the local holomorphic coordinates on $M$ and $(z^\alpha, v^\alpha)$ be the induced holomorphic coordinates on the holomorphic tangent bundle $T^{1,0}M$. Assume that $M$ is endowed with a strongly pseudoconvex complex Finsler metric $F$, i.e., a non-negative function $F : T^{1,0}M \to \mathbb{R}^+$ satisfying [1, p. 84]

(1) $F(z, v)$ is smooth on $\tilde{M} = T^{1,0}M/\{0\}$;
(2) $F(z, v) > 0$ for all $(z, v) \in \tilde{M}$ and $F(z, v) = 0$ if and only if $v = 0$;
(3) $F(z, \lambda v) = |\lambda|^2 F(z, v)$ for all $(z, v) \in T^{1,0}M$ and $\lambda \in \mathbb{C}$;
(4) the Hermitian matrix

$$(\tilde{g}_{\alpha\bar{\beta}}(z, v)) = \left(\frac{\partial^2 F}{\partial v^\alpha \partial \bar{v}^\beta}\right) \tag{1.1}$$

is positive definite on $\tilde{M}$.

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*E-mail address:* zcp@xmu.edu.cn.

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Now, let $f : \mathcal{M} \to M$ be a holomorphic immersion of an $m$-dimensional complex manifold $\mathcal{M}$ into $M$, which is locally given by the equations

$$z^\alpha = z^\alpha(w^1, \dots, w^m), \quad \operatorname{rank}\left(\frac{\partial z^\alpha}{\partial w^i}\right) = m.$$

If we denote by $(w^i)$ the local holomorphic coordinates on $\mathcal{M}$ and $(w^i, \eta^i)$ the induced holomorphic coordinates on $T^{1,0}\mathcal{M}$, then the differential $(f_*)_w : T_w^{1,0}\mathcal{M} \to T_{f(w)}^{1,0}M$ is injective for every point $w \in \mathcal{M}$ and locally one may assume that $\mathcal{M}$ is imbedded in $M$. Since a point $(w, \eta) \in T^{1,0}\mathcal{M}$ is carried by $(f_*)_w$ into a point $(f(w), f_*\eta) \in T^{1,0}M$ with

$$v^\alpha = \eta^i B_i^\alpha, \quad \text{where } B_i^\alpha = \frac{\partial z^\alpha}{\partial w^i}, \tag{1.2}$$

the induced complex Finsler metric $\mathcal{F} : T^{1,0}\mathcal{M} \to \mathbb{R}^+$ is given by

$$\mathcal{F}(w, \eta) = F(f(w), f_*\eta) \tag{1.3}$$

with induced Finsler tensor field

$$g_{i\bar{j}}(w, \eta) = \tilde{g}_{\alpha\bar{\beta}}(f(w), f_*\eta) B_i^\alpha B_{\bar{j}}^{\bar{\beta}}, \tag{1.4}$$

where we have denote by $B_{\bar{j}}^{\bar{\beta}}$ the complex conjugation of $B_j^\beta$.

Throughout this paper we shall use the following range of indices: $\alpha, \beta, \gamma, \mu, \sigma, \tau, \dots \in \{1, \dots, n\}$; $i, j, k, l, s \dots \in \{1, \dots, m\}$; $a, b, c, d, \dots \in \{m+1, \dots, n\}$, and the Einstein sum convention is assumed.

Obviously, $\mathcal{F}$ is a strongly pseudoconvex complex Finsler metric on $\mathcal{M}$. We call the pair $(\mathcal{M}, \mathcal{F})$ a strongly pseudoconvex complex Finsler submanifold of $(M, F)$. There are two kinds of connection in Finsler submanifolds, that is, the intrinsic and the induced connections. The former is due to the fundamental function $\mathcal{F}$ of the submanifold $\mathcal{M}$ and the latter is the induced one on $\mathcal{M}$ from the given Finsler connection on $(M, F)$ and these connections are different from each other in general real Finsler manifolds, which contrast sharply distinction to the Riemannian case. However G. Munteanu [8, p. 137, Corollary 5.4.2] shows that the induced connection on $\mathcal{M}$ coincides with the intrinsic Chern–Finsler connection on $(\mathcal{M}, \mathcal{F})$ when $(M, F)$ is endowed with the canonical Chern–Finsler connection.

The aim of this paper is to investigate the geometry of the complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ that holomorphically immersed in $(M, F)$. We first establish the fundamental equations, i.e., Gauss, Codazzi and Ricci equations of $(\mathcal{M}, \mathcal{F})$. Then as application we prove that the holomorphic curvature of $(\mathcal{M}, \mathcal{F})$ is not exceed that of $(M, F)$ and they coincide if and only if the horizontal components of the second fundamental form of $(\mathcal{M}, \mathcal{F})$ vanish identically. We also prove that a given complex Finsler submanifold of a weakly Kähler Finsler manifold is totally geodesic if and only if a suitable contraction of the horizontal components of the second fundamental form of $(\mathcal{M}, \mathcal{F})$ vanish. This shows the importance of the horizontal fundamental form of complex Finsler submanifolds in the theory of complex Finsler submanifolds. See Theorem 6.1 and Corollary 6.4 in Section 6 for details.

For a general theory of submanifolds in a real Finsler manifold, see [3–5,9] and the references therein.

This paper is arranged as follows. In Section 1 we recall some notations and basic results for the vectorial Finsler connection, which is widely used in investigating the geometry of submanifolds in real Finsler geometry [3–5]. In Section 2, we recall some results and notations about the Chern–Finsler connection on a strongly pseudoconvex complex Finsler manifold, which will be used in later sections. We refer to [1] for more details of the Chern–Finsler connection on strongly pseudoconvex complex Finsler manifolds. In Section 3, we deal with the induced complex linear connection and normal Finsler connection that induced by the ambient complex Finsler manifold endowed with the canonical Chern–Finsler connection. We first derive the coefficients of the induced linear connection and the normal Finsler connection in terms of the coefficients of the Chern–Finsler connection on the ambient manifold. Then we investigate the $h(v)$-relative covariant derivatives of some mixed tensor fields which will be used in section Section 4. In Section 4, we derive locally the fundamental equations for the complex Finsler submanifolds, i.e., the Gauss, Codazzi and Ricci equations for the complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ with respect to the Chern–Finsler connection on $(M, F)$. In Section 5, we investigate the relationship between the torsion of the complex linear connection that associated to the induced Chern–Finsler connection on $(\mathcal{M}, \mathcal{F})$ and the torsion of the complex linear connection that associated to the Chern–Finsler connection on the ambient manifold $(M, F)$. As an application of the fundamental formulas of the complex Finsler submanifolds we prove that the holomorphic curvature of the induced complex Finsler metric $\mathcal{F}$ does not exceed the holomorphic curvature of $F$. Furthermore we give a characterization of the totally geodesic complex Finsler submanifold in terms of the horizontal fundamental form of $(\mathcal{M}, \mathcal{F})$.

Now, we denote by $\mathrm{d}\pi : T^{1,0}\tilde{M} \to \tilde{M}$ the differential of $\pi$. Since the projection $\pi$ restricted to the slit holomorphic tangent bundle $\tilde{M}$ is holomorphic, one can define the (complex) vertical bundle $\mathcal{V}(\tilde{M})$ of $T^{1,0}\tilde{M}$ as $\ker d\pi \subset T^{1,0}\tilde{M}$. The vertical vector bundle is a holomorphic vector bundle over $\tilde{M}$ which is locally spanned by $\{\frac{\partial}{\partial v^\alpha}\}$. A complementary distribution $\mathcal{H}(\tilde{M})$ to $\mathcal{V}(\tilde{M})$ in $T^{1,0}\tilde{M}$ is called a (complex) horizontal subbundle of $T^{1,0}\tilde{M}$, or called the nonlinear connection on $\tilde{M}$. Locally, if we consider a canonical holomorphic chart $(U, \varphi)$ on $\tilde{M}$, then the complex nonlinear connection $\mathcal{H}(\tilde{M})$ is determined by $n^2$ complex valued differentiable functions $N_\alpha^\beta(z, v)$ on each $U$ satisfying

$$N_{\alpha'}^{\beta'} = \frac{\partial z^{\beta'}}{\partial z^\gamma}\frac{\partial z^\delta}{\partial z^{\alpha'}}N_\delta^\gamma - \frac{\partial^2 z^{\beta'}}{\partial z^\gamma \partial z^\delta}\frac{\partial z^\gamma}{\partial z^{\alpha'}}v^\delta, \tag{1.5}$$

where $N_{\alpha'}^{\beta'}$ are the corresponding functions on the domain $U'$ of another holomorphic chart $(U', \varphi')$ with $U \cap U' \neq \varnothing$. Moreover, if $\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial v^\alpha}\}$ is the natural local frame on $\tilde{M}$ then $\{\frac{\delta}{\delta z^\alpha}\}$ given by

$$\frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} - N_\alpha^\beta \frac{\partial}{\partial v^\beta}, \quad \alpha = 1, \ldots, n, \tag{1.6}$$

is a local frame of the distribution $\mathcal{H}(\tilde{M})$. The concept of complex nonlinear connection plays an important role in complex Finsler geometry. It takes an adapted local frame instead of the natural local frame in order to linearize the geometry of the holomorphic tangent bundle.

Next let $\pi_E : E \to \tilde{M}$ be a holomorphic vector bundle of rank $r$ over the slit holomorphic tangent bundle $\tilde{M}$. Denote by $\mathcal{X}(E)$ and $\mathcal{X}(T^{1,0}\tilde{M})$ the module of differentiable sections of $E$ and respectively $T^{1,0}\tilde{M}$. Suppose that $D : \mathcal{X}(E) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{M} \otimes E)$ is a Hermitian connection such that

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_{\overline{X}} Z\rangle$$

for all $X \in T^{1,0}\tilde{M}$ and $Y, Z \in \mathcal{X}(E)$.

The pair $VFC = (\mathcal{H}(\tilde{M}), D)$ with a complex nonlinear connection $\mathcal{H}(\tilde{M})$ on $\tilde{M}$ and a Hermitian connection $D$ on a holomorphic vector bundle $E$ over $\tilde{M}$ is called a vectorial Finsler connection on $E$. Vectorial Finsler connection is widely used in the investigation of the geometry of submanifolds in real Finsler geometry [5]. The method of vectorial Finsler connection is also widely used in complex Finsler geometry, we refer to [1,2,6–8] for an introduction to the complex Finsler geometry. However, the Cartan connection and the Chern–Finsler connection that associated to a given strongly convex complex Finsler metric are less related than in the Hermitian case even for a Kähler–Finsler metric. We refer to [1] for a detailed comparison of the Cartan connection and the Chern–Finsler connection that associated to a strongly convex complex Finsler metric. Therefore it seems naturally to develop the theory of complex submanifold of a given complex Finsler manifold by means of the Chern–Finsler connection. This motivate us to investigate the geometry of complex submanifolds of a complex Finsler manifold via the idea of vectorial Finsler connection in complex settings.

If we denote by

$$B_{ij}^\alpha = \frac{\partial^2 z^\alpha}{\partial w^i \partial w^j}, \quad B_{0j}^\alpha = \eta^i B_{ij}^\alpha,$$

then the natural local frame $\{\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \eta^i}\}$ and its dual frame $\{dw^i, d\eta^i\}$ on $\tilde{\mathcal{M}}$ and the natural local frame $\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial v^\alpha}\}$ and its dual frame $\{dz^\alpha, dv^\alpha\}$ on $\tilde{M}$ are related by

$$\frac{\partial}{\partial w^i} = B_i^\alpha \frac{\partial}{\partial z^\alpha} + B_{0i}^\alpha \frac{\partial}{\partial v^\alpha}, \tag{1.7}$$

$$\frac{\partial}{\partial \eta^i} = B_i^\alpha \frac{\partial}{\partial v^\alpha}, \tag{1.8}$$

$$dz^\alpha = B_i^\alpha dw^i, \tag{1.9}$$

$$dv^\alpha = B_{0j}^\alpha d\eta^j + B_j^\alpha d\eta^j. \tag{1.10}$$

Note that the Finsler fundamental tensor $\tilde{g}_{\alpha\bar{\beta}}(z, v)$ defines a Hermitian metric $\tilde{g}$ on the holomorphic vertical subbundle $\mathcal{V}(\tilde{M})$ of $T^{1,0}\tilde{M}$. If we denote by $\mathcal{V}(\tilde{M}^*)$ the restriction of the holomorphic vertical subbundle $\mathcal{V}(\tilde{M})$ to the points of $\tilde{\mathcal{M}}$, then $\mathcal{V}(\tilde{M}^*)$ is obviously a holomorphic vector bundle of rank $n$ over $\tilde{\mathcal{M}}$.

On the other hand by (1.8) we see that $\mathcal{V}(\tilde{\mathcal{M}})$ is a holomorphic vector subbundle of $\mathcal{V}(\tilde{M}^*)$ and we have

$$g_{i\bar{j}} = \tilde{g}\left(\frac{\partial}{\partial\eta^i}, \frac{\partial}{\partial\eta^j}\right), \tag{1.11}$$

i.e., $\mathcal{V}(\tilde{\mathcal{M}})$ is a holomorphic vector subbundle of $\mathcal{V}(\tilde{M}^*)$ with the induced Hermitian metric $g$.

In order to define the local components of some induced geometric objects on $(\mathcal{M}, \mathcal{F})$ we consider a local orthonormal frame $\{B_a = B_a^\gamma \frac{\partial}{\partial v^\gamma}\}$ in $\mathcal{V}(\tilde{\mathcal{M}})^\perp$ with respect to $\tilde{g}$. That is,

$$\tilde{g}\left(\frac{\partial}{\partial\eta^i}, B_a\right) = \tilde{g}_{\alpha\bar{\beta}} B_i^\alpha B_{\bar{a}}^{\bar{\beta}} = 0, \tag{1.12}$$

$$\tilde{g}(B_a, B_b) = \tilde{g}_{\alpha\bar{\beta}} B_a^\alpha B_{\bar{b}}^{\bar{\beta}} = \delta_{a\bar{b}}. \tag{1.13}$$

The local basis $\{\frac{\partial}{\partial\eta^i} = B_i^\alpha \frac{\partial}{\partial v^\alpha}, B_a = B_a^\alpha \frac{\partial}{\partial v^\alpha}\}$ in $\mathcal{V}(\tilde{M}^*)$ is called the Finsler fields of frame on $(M, F)$ along $(\mathcal{M}, \mathcal{F})$. Denote by $(B_i^\alpha B_a^\alpha)$ the transition matrix from the natural fields of frame $\{\frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\}$ on $\mathcal{V}(\tilde{M}^*)$ to the Finsler field of frames $\{B_i^\alpha \frac{\partial}{\partial v^\alpha}, B_a^\alpha \frac{\partial}{\partial v^\alpha}\}$. Let $(\mathcal{B}_\alpha^i \mathcal{B}_\alpha^a)$ be the inverse matrix of $(B_i^\alpha B_a^\alpha)$, then we have

$$\mathcal{B}_\alpha^i B_j^\alpha = \delta_j^i; \qquad \mathcal{B}_\alpha^i B_a^\alpha = 0; \qquad \mathcal{B}_\alpha^a B_i^\alpha = 0; \qquad \mathcal{B}_\alpha^a B_b^\alpha = \delta_b^a; \tag{1.14}$$

and

$$B_i^\alpha \mathcal{B}_\beta^i + B_a^\alpha \mathcal{B}_\beta^a = \delta_\beta^\alpha. \tag{1.15}$$

It is easy to check that

$$\tilde{g}^{\bar{\beta}\alpha} \mathcal{B}_\alpha^a \mathcal{B}_{\bar{\beta}}^{\bar{i}} = 0, \tilde{g}^{\bar{\beta}\alpha} \mathcal{B}_\alpha^i \mathcal{B}_{\bar{\beta}}^{\bar{j}} = g^{\bar{j}i}. \tag{1.16}$$

Now contracting (1.4) by $g^{\bar{j}l}\mathcal{B}_\sigma^i$ and taking into account (1.15) and (1.12) we deduce that

$$\mathcal{B}_\sigma^l = \tilde{g}_{\sigma\bar{\beta}} B_{\bar{j}}^{\bar{\beta}} g^{\bar{j}l}. \tag{1.17}$$

## 2. Chern–Finsler connection

In this section we recall the Chern–Finsler connection on the strongly pseudoconvex complex Finsler manifold $(M, F)$, we refer to [1] for more details.

Let $D : \mathcal{X}(\mathcal{V}(\tilde{M})) \to \mathcal{X}(T_{\mathbb{C}}^*\tilde{M} \otimes \mathcal{V}(\tilde{M}))$ be the Hermitian connection in $(\mathcal{V}(\tilde{M}), \tilde{g})$, which is also called the Chern–Finsler connection associated to $(M, F)$ in [1, p. 87]. Locally it is characterized by a triple $(N_\mu^\alpha, F_{\beta;\mu}^\alpha, F_{\beta\mu}^\alpha)$, where

$$N_\mu^\alpha = \tilde{g}^{\bar{\tau}\alpha} \frac{\partial^2 F}{\partial z^\mu \partial\bar{v}^\tau}, \tag{2.1}$$

$$F_{\beta;\mu}^\alpha = \tilde{g}^{\bar{\tau}\alpha} \frac{\delta\tilde{g}_{\beta\bar{\tau}}}{\delta z^\mu}, \tag{2.2}$$

$$F_{\beta\mu}^\alpha = \tilde{g}^{\bar{\tau}\alpha} \frac{\partial\tilde{g}_{\beta\bar{\tau}}}{\partial v^\mu}, \tag{2.3}$$

and $\frac{\delta}{\delta z^\mu}$ is given by (1.6) with $N_\mu^\alpha$ being defined by (2.1).

Let $\mathcal{H}(\tilde{M})$ be the complex horizontal distribution that associated to the Chern–Finsler connection. We have the following decompositions of vector bundles:

$$T_{\mathbb{C}}\tilde{M} = \mathcal{H}_{\mathbb{C}}(\tilde{M}) \oplus \mathcal{V}_{\mathbb{C}}(\tilde{M}), \tag{2.4}$$

where

$$\mathcal{H}_{\mathbb{C}}(\tilde{M}) = \mathcal{H}(\tilde{M}) \oplus \overline{\mathcal{H}(\tilde{M})}, \qquad \mathcal{V}_{\mathbb{C}}(\tilde{M}) = \mathcal{V}(\tilde{M}) \oplus \overline{\mathcal{V}(\tilde{M})}$$

and

$$T^{1,0}\tilde{M} = \mathcal{H}(\tilde{M}) \oplus \mathcal{V}(\tilde{M}), \qquad T^{0,1}\tilde{M} = \overline{\mathcal{H}(\tilde{M})} \oplus \overline{\mathcal{V}(\tilde{M})}.$$

Now denote by $\tilde{v}$ and $\tilde{h}$ the projections of $T_{\mathbb{C}}\tilde{M}$ to $\mathcal{H}_{\mathbb{C}}(\tilde{M})$ and respectively $\mathcal{V}_{\mathbb{C}}(\tilde{M})$ with respect to (2.4). Moreover, let $\tilde{Q}$ be the almost product structure that associated to $\mathcal{H}(\tilde{M})$. More precisely, if $X = X^\alpha \frac{\delta}{\delta z^\alpha} + \dot{X}^\alpha \frac{\partial}{\partial v^\alpha} + Y^\beta \frac{\delta}{\delta \bar{z}^\beta} + \dot{Y}^\beta \frac{\partial}{\partial \bar{v}^\beta} \in T_{\mathbb{C}}\tilde{M}$, then

$$\tilde{Q}(X) = \dot{X}^\alpha \frac{\delta}{\delta z^\alpha} + X^\alpha \frac{\partial}{\partial v^\alpha} + \dot{Y}^\beta \frac{\delta}{\delta \bar{z}^\beta} + Y^\beta \frac{\partial}{\partial \bar{v}^\beta}. \tag{2.5}$$

The almost product structure $\tilde{Q}$ satisfies

$$\tilde{Q} \circ \tilde{v} = \tilde{h} \circ \tilde{Q}, \qquad \tilde{Q} \circ \tilde{h} = \tilde{v} \circ \tilde{Q}, \qquad \tilde{Q}^2 = I, \tag{2.6}$$

where $I$ is the identity operator on $T_{\mathbb{C}}\tilde{M}$. Using the almost product structure $\tilde{Q}$, one can extend the Chern–Finsler connection to a complex linear connection (still denote by $D$) $D : \mathcal{X}(T_{\mathbb{C}}\tilde{M}) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{M} \otimes T_{\mathbb{C}}\tilde{M})$. Since the Chern–Finsler connection is a Hermitian connection, the extended complex linear connection $D$ on $T_{\mathbb{C}}\tilde{M}$ can be expressed as follows:

$$D_X Y = D_X \tilde{v} Y + \tilde{Q}[D_X \tilde{Q}\tilde{h} Y], \quad \forall X \in T_{\mathbb{C}}\tilde{M}, Y \in \mathcal{X}(T^{1,0}\tilde{M}). \tag{2.7}$$

The Chern–Finsler connection is both $h$-metrical and $v$-metrical. Moreover, the nonlinear connection coefficients $N^\alpha_\mu$ and the horizontal connection coefficients $F^\alpha_{\beta;\mu}$ are related by

$$F^\alpha_{\beta;\mu} v^\beta = N^\alpha_\mu, \qquad F^\alpha_{\beta;\mu} = \frac{\partial N^\alpha_\mu}{\partial v^\beta}. \tag{2.8}$$

The vertical connection coefficients $F^\alpha_{\beta\mu}$ satisfy

$$F^\alpha_{\beta\mu} = F^\alpha_{\mu\beta}, \quad F^\alpha_{\beta\mu} v^\mu = F^\alpha_{\beta\mu} v^\beta = 0. \tag{2.9}$$

The Lie bracket of the horizontal frame $\{\frac{\delta}{\delta z^\alpha}\}$ of $\mathcal{H}(\tilde{M})$ has the following useful properties.

**Proposition 2.1** (*[1, p. 89, Lemma 2.3.3]*). *Let $D$ be the Chern–Finsler connection associated to a strongly pseudoconvex complex Finsler metric $F$, and let $\{\frac{\delta}{\delta z^\alpha}\}$ the corresponding local horizontal frame. Then*

$$\left[\frac{\delta}{\delta z^\mu}, \frac{\delta}{\delta z^\nu}\right] = 0, \qquad \left[\frac{\delta}{\delta z^\mu}, \frac{\partial}{\partial v^\alpha}\right] = F^\sigma_{\alpha;\mu} \frac{\partial}{\partial v^\sigma}, \qquad \left[\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right] = 0, \tag{2.10}$$

$$\left[\frac{\delta}{\delta z^\mu}, \frac{\delta}{\delta \bar{z}^\nu}\right] = \frac{\delta}{\delta \bar{z}^\nu}(N^\sigma_\mu) \frac{\partial}{\partial v^\sigma} - \frac{\delta}{\delta z^\mu}(N^{\bar{\tau}}_{\bar{\nu}}) \frac{\partial}{\partial \bar{v}^\tau}, \tag{2.11}$$

$$\left[\frac{\delta}{\delta z^\mu}, \frac{\partial}{\partial \bar{v}^\alpha}\right] = \frac{\partial}{\partial \bar{v}^\alpha}(N^\sigma_\mu) \frac{\partial}{\partial v^\sigma}, \qquad \left[\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial \bar{v}^\beta}\right] = 0. \tag{2.12}$$

Next we denote by $T(X, Y) = D_X Y - D_Y X - [X, Y]$ the torsion of the complex linear connection $D$, by using (2.7) we obtain

$$T(X, Y) = (D_X \tilde{v} Y - D_Y \tilde{v} X - \tilde{v}[X, Y]) + \tilde{Q}(D_X \tilde{Q}\tilde{h} Y - D_Y \tilde{Q}\tilde{h} X - \tilde{Q}\tilde{h}[X, Y]) \tag{2.13}$$

for any $X, Y \in T_{\mathbb{C}}\tilde{M}$. Locally, we derive that

$$T\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta z^\beta}\right) = T^\gamma_{;\beta\alpha}\frac{\delta}{\delta z^\gamma}, \qquad T\left(\frac{\partial}{\partial v^\alpha}, \frac{\delta}{\delta z^\beta}\right) = T^\gamma_{\alpha;\beta}\frac{\delta}{\delta z^\gamma}, \tag{2.14}$$

$$T\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}\right) = \dot{T}^\gamma_{;\alpha\bar{\beta}}\frac{\partial}{\partial v^\gamma} + \dot{T}^{\bar{\gamma}}_{;\alpha\bar{\beta}}\frac{\partial}{\partial \bar{v}^\gamma}, \qquad T\left(\frac{\delta}{\delta z^\alpha}, \frac{\partial}{\partial \bar{v}^\beta}\right) = \dot{T}^\gamma_{\bar{\beta};\alpha}\frac{\partial}{\partial v^\gamma}, \tag{2.15}$$

where we have denoted by

$$T^\gamma_{;\beta\alpha} = F^\gamma_{\beta;\alpha} - F^\gamma_{\alpha;\beta}, \qquad T^\gamma_{\alpha;\beta} = F^\gamma_{\beta\alpha}, \tag{2.16}$$

$$\dot{T}^\gamma_{;\alpha\bar{\beta}} = -\frac{\delta}{\delta \bar{z}^\beta}(N^\gamma_\alpha), \qquad \dot{T}^{\bar{\gamma}}_{;\alpha\bar{\beta}} = \frac{\delta}{\delta z^\alpha}(N^{\bar{\gamma}}_{\bar{\beta}}), \quad \dot{T}^\gamma_{\bar{\beta};\alpha} = -\frac{\partial}{\partial \bar{v}^\beta}(N^\gamma_\alpha). \tag{2.17}$$

Obviously we have $\overline{\dot{T}^{\bar{\gamma}}_{;\alpha\bar{\beta}}} = -\dot{T}^\gamma_{;\beta\bar{\alpha}}$. These are all the non-zero components of the torsion Finsler tensor field $T$ of the complex linear connection $D$ that associated to the Chern–Finsler connection on $(M, F)$.

Now let us denote by $\tilde{R}$ the curvature tensor of the complex linear connection $D$. Using (2.6) we derive that for all $X, Y, Z \in T^{1,0}\tilde{M}$,

$$\tilde{R}(X, Y)Z = \tilde{\Omega}(X, Y)\tilde{v}Z + \tilde{Q}\tilde{\Omega}(X, Y)\tilde{Q}\tilde{h}Z, \tag{2.18}$$

$$\tilde{R}(X, \overline{Y})Z = \tilde{\Omega}(X, \overline{Y})\tilde{v}Z + \tilde{Q}\tilde{\Omega}(X, \overline{Y})\tilde{Q}\tilde{h}Z, \tag{2.19}$$

where $\tilde{\Omega}(X, Y)$, $\tilde{\Omega}(X, \overline{Y})$ denote the curvatures of the Chern–Finsler connection. Note that $\tilde{R}(X, Y)Z \equiv 0$ since the $(2, 0)$-form of the curvature of the Chern–Finsler connection vanishes identically. If we put

$$\tilde{R}\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}\right)\frac{\partial}{\partial v^\gamma} = \tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}}\frac{\partial}{\partial v^\sigma}, \qquad \tilde{R}\left(\frac{\partial}{\partial v^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}\right)\frac{\partial}{\partial v^\gamma} = \tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}}\frac{\partial}{\partial v^\sigma}, \tag{2.20}$$

$$\tilde{R}\left(\frac{\delta}{\delta z^\alpha}, \frac{\partial}{\partial \bar{v}^\beta}\right)\frac{\partial}{\partial v^\gamma} = \tilde{R}^\sigma_{\gamma\bar{\beta};\alpha}\frac{\partial}{\partial v^\sigma}, \qquad \tilde{R}\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial \bar{v}^\beta}\right)\frac{\partial}{\partial v^\gamma} = \tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}\frac{\partial}{\partial v^\sigma}, \tag{2.21}$$

then by (2.19), we have

$$\tilde{R}\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}\right)\frac{\delta}{\delta z^\gamma} = \tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}}\frac{\delta}{\delta z^\sigma}, \qquad \tilde{R}\left(\frac{\partial}{\partial v^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}\right)\frac{\delta}{\delta z^\gamma} = \tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}}\frac{\delta}{\delta z^\sigma}, \tag{2.22}$$

$$\tilde{R}\left(\frac{\delta}{\delta z^\alpha}, \frac{\partial}{\partial \bar{v}^\beta}\right)\frac{\delta}{\delta z^\gamma} = \tilde{R}^\sigma_{\gamma\bar{\beta};\alpha}\frac{\delta}{\delta z^\sigma}, \qquad \tilde{R}\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial \bar{v}^\beta}\right)\frac{\delta}{\delta z^\gamma} = \tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}\frac{\delta}{\delta z^\sigma}, \tag{2.23}$$

where

$$\tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}} = -\frac{\delta}{\delta \bar{z}^\beta}(F^\sigma_{\gamma;\alpha}) - F^\sigma_{\gamma\tau}\frac{\delta}{\delta \bar{z}^\beta}(N^\tau_\alpha), \tag{2.24}$$

$$\tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}} = -\frac{\delta}{\delta \bar{z}^\beta}(F^\sigma_{\gamma\alpha}), \tag{2.25}$$

$$\tilde{R}^\sigma_{\gamma\bar{\beta};\alpha} = -\frac{\partial}{\partial \bar{v}^\beta}(F^\sigma_{\gamma;\alpha}) - F^\sigma_{\gamma\tau}\frac{\partial}{\partial \bar{v}^\beta}(N^\tau_\alpha), \tag{2.26}$$

$$\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}} = -\frac{\partial}{\partial \bar{v}^\beta}(F^\sigma_{\gamma\alpha}). \tag{2.27}$$

These are the only non-zero curvature components of $\tilde{R}$ of the Chern–Finsler connection.

## 3. Induced Finsler connections by *CFC* on $(M, F)$

Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection $(N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$. In this section we will derive various kinds of induced complex linear connections. First we have

**Theorem 3.1.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection $D$ and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$. Then there exists a unique complex horizontal distribution $\mathcal{H}(\tilde{\mathcal{M}})$ on $\tilde{\mathcal{M}}$ satisfying*

$$\mathcal{H}(\tilde{\mathcal{M}}) \subset \mathcal{V}(\tilde{\mathcal{M}})^{\perp} \oplus \mathcal{H}(\tilde{M}^*), \tag{3.1}$$

*where $\mathcal{H}(\tilde{M}^*)$ is the restriction of $\mathcal{H}(\tilde{M})$ to $\tilde{\mathcal{M}}$.*

**Proof.** If we define locally the complex valued differentiable functions

$$\mathcal{N}^i_j = \mathcal{B}^i_\alpha (B^\alpha_{0j} + N^\alpha_\beta B^\beta_j). \tag{3.2}$$

Then it is easy to check that $\mathcal{N}^i_j$ satisfies the transformation law of (1.5). Thus we obtain a complex horizontal distribution $\mathcal{H}(\tilde{\mathcal{M}})$ on $\tilde{\mathcal{M}}$, which is locally spanned by

$$\frac{\delta}{\delta w^j} = \frac{\partial}{\partial w^j} - \mathcal{N}^i_j \frac{\partial}{\partial \eta^i}, \quad j = 1, \ldots, m, \tag{3.3}$$

where $\mathcal{N}^i_j$ is given by (3.2). By (1.8) we have

$$\frac{\partial}{\partial v^\alpha} = \mathcal{B}^i_\alpha \frac{\partial}{\partial \eta^i} + \mathcal{B}^a_\alpha B_a. \tag{3.4}$$

Using (1.6) and (3.2)–(3.4), we derive that

$$\frac{\delta}{\delta w^i} = B^\alpha_i \frac{\delta}{\delta z^\alpha} + H^a_i B_a, \tag{3.5}$$

where $H^a_i$ are given by

$$H^a_i = \mathcal{B}^a_\alpha (B^\alpha_{0i} + N^\alpha_\beta B^\beta_i). \tag{3.6}$$

Thus by (3.3) we obtain the existence of $\mathcal{H}(\tilde{\mathcal{M}})$ satisfying (3.1).

Suppose that $\mathcal{N}'^i_j$ is another complex horizontal distribution on $\tilde{\mathcal{M}}$ that satisfying (3.1), that is, (3.5) with $H^a_i$ as arbitrary complex functions. Using (1.6), (1.7) and (3.5) we have

$$\frac{\delta}{\delta w^i} = B^\alpha_i \frac{\delta}{\delta z^\alpha} + (N^\beta_\alpha B^\alpha_i + B^\beta_{0i} - \mathcal{N}'^j_i B^\beta_j) \frac{\partial}{\partial v^\beta}. \tag{3.7}$$

Now it follows from (3.7) and (3.5) that

$$H^a_i B^\beta_a = N^\beta_\alpha B^\alpha_i + B^\beta_{0i} - \mathcal{N}'^j_i B^\beta_j. \tag{3.8}$$

Contracting (3.8) with $\mathcal{B}^i_\beta$ we have $\mathcal{N}'^j_i = \mathcal{N}^j_i$ given by (3.2) and contracting (3.8) with $\mathcal{B}^b_\beta$ we get (3.6). Thus we complete the proof of the uniqueness of the complex horizontal distribution given by (3.1). ■

The complex horizontal distribution $\mathcal{H}(\tilde{\mathcal{M}})$ given by (3.2) is called the induced complex horizontal distribution by $\mathcal{H}(\tilde{M})$ on $\tilde{\mathcal{M}}$.

In the following we denote $\{\delta w^i, \frac{\partial}{\partial \eta^i}\}$ the adapted frame on $T^{1,0}\tilde{\mathcal{M}}$ and $\{dw^i, \delta\eta^i\}$ its dual frame, where $\delta\eta^i = d\eta^i + \mathcal{N}^i_j dw^j$. It is easy to check that

$$\delta v^\alpha = B^\alpha_i \delta\eta^i + B^\alpha_a H^a_i dw^i. \tag{3.9}$$

Since $\mathcal{V}(\tilde{M}^*)$ is a holomorphic subbundle of $\mathcal{V}(\tilde{M})$. We denote by $D^* : \mathcal{X}(\mathcal{V}(\tilde{M}^*)) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{\mathcal{M}} \otimes \mathcal{V}(\tilde{M}^*))$ the restriction of the Chern–Finsler connection $D : \mathcal{X}(\mathcal{V}(\tilde{M})) \to \mathcal{X}(T^*_{\mathbb{C}}\tilde{M} \otimes \mathcal{V}(\tilde{M}))$ to the holomorphic subbundle $\mathcal{V}(\tilde{M}^*)$. $D^*$ is obviously a Hermitian connection of type (1, 0). Let

$$D^*_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial v^\alpha} = D_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial v^\alpha} = F^{*\gamma}_{\alpha;k} \frac{\partial}{\partial v^\gamma}, \tag{3.10}$$

$$D^*_{\frac{\partial}{\partial \eta^k}} \frac{\partial}{\partial v^\alpha} = D_{\frac{\partial}{\partial \eta^k}} \frac{\partial}{\partial v^\alpha} = F^{*\gamma}_{\alpha k} \frac{\partial}{\partial v^\gamma}. \tag{3.11}$$

Expressing the connection coefficients $F^{*\gamma}_{\alpha;k}$, $F^{*\gamma}_{\alpha k}$ in terms of the connection coefficients $F^{\alpha}_{\beta;\mu}$, $F^{\alpha}_{\beta\mu}$, we derive that

$$F^{*\gamma}_{\alpha;k} = F^{\gamma}_{\alpha;\beta} B^{\beta}_{k} + F^{\gamma}_{\alpha\beta} B^{\beta}_{a} H^{a}_{k}, \tag{3.12}$$

$$F^{*\gamma}_{\alpha k} = F^{\gamma}_{\alpha\beta} B^{\beta}_{k}. \tag{3.13}$$

Thus we have a vectorial Finsler connection $IVFC = (\mathcal{N}^i_j, F^{*\gamma}_{\alpha;i}, F^{*\gamma}_{\alpha i})$.

On the other hand, $\mathcal{V}(\tilde{\mathcal{M}})$ is a holomorphic subbundle of $\mathcal{V}(\tilde{M}^*)$. Thus the quotient bundle $\mathcal{Q} = \mathcal{V}(\tilde{M}^*)/\mathcal{V}(\tilde{\mathcal{M}})$ is a holomorphic vector subbundle of rank $(n-m)$. One can express this as an exact sequence

$$0 \to \mathcal{V}(\tilde{\mathcal{M}}) \to \mathcal{V}(\tilde{M}^*) \to \mathcal{Q} \to 0.$$

Taking the orthogonal complement of $\mathcal{V}(\tilde{\mathcal{M}})$ in $\mathcal{V}(\tilde{M}^*)$ with respect to $\tilde{g}$, we obtain a complex subbundle $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$ of $\mathcal{V}(\tilde{M}^*)$. Note that $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$ may not be a holomorphic subbundle of $\mathcal{V}(\tilde{M}^*)$ in general. Thus

$$\mathcal{V}(\tilde{M}^*) = \mathcal{V}(\tilde{\mathcal{M}}) \oplus \mathcal{V}(\tilde{\mathcal{M}})^{\perp} \tag{3.14}$$

is merely a $C^\infty$ orthogonal decomposition of $\mathcal{V}(\tilde{M}^*)$. As a $C^\infty$ complex vector bundle, $\mathcal{Q}$ is naturally isomorphic to $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$. Hence we have an induced Hermitian structure on $\mathcal{Q}$ in a natural way.

In the following we denote by $g$ the restriction of the Hermitian metric $\tilde{g}$ to $\mathcal{V}(\tilde{\mathcal{M}})$. Now we define $\mathcal{D}$ and $B$ by

$$D^*_X Y = \mathcal{D}_X Y + B(X, Y), \quad \forall X \in T^{1,0}\tilde{\mathcal{M}}, Y \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}})). \tag{3.15}$$

**Proposition 3.2.** (1) *$\mathcal{D}$ is the Hermitian connection of $(\mathcal{V}(\tilde{\mathcal{M}}), g)$. (2) $B$ is a $(1,0)$-form with values in* $\mathrm{Hom}(\mathcal{V}(\tilde{\mathcal{M}}), \mathcal{V}(\tilde{\mathcal{M}})^{\perp})$.

**Proof.** Let $f$ be a function on $\tilde{\mathcal{M}}$. Replacing $Y$ by $fY$ in (3.15) we obtain

$$D^*_X(fY) = \mathcal{D}_X(fY) + B(X, fY).$$

On the other hand,

$$D^*_X(fY) = df(X)\cdot Y + fD^*_X Y = df(X)\cdot Y + f\mathcal{D}_X Y + fB(X, Y).$$

Comparing the $\mathcal{V}(\tilde{\mathcal{M}})$- and $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$-components of the two decompositions of $D^*(fY)$ we conclude

$$\mathcal{D}_X(fY) = df(X)\cdot Y + f\mathcal{D}_X Y, \qquad B(X, fY) = fB(X, Y).$$

The first equality says that $\mathcal{D}$ is a connection and the second says that $B$ is a 1-form with values in $\mathrm{Hom}(\mathcal{V}(\tilde{\mathcal{M}}), \mathcal{V}(\tilde{\mathcal{M}})^{\perp})$. If $Y$ in (3.15) is holomorphic, then $D^*Y$ is a $(1,0)$-form with values in $\mathcal{V}(\tilde{M}^*)$, hence, $\mathcal{D}Y$ is a $(1,0)$-form with values in $\mathcal{V}(\tilde{\mathcal{M}})$ while $B$ is a $(1,0)$-form with values in $\mathrm{Hom}(\mathcal{V}(\tilde{\mathcal{M}}), \mathcal{V}(\tilde{\mathcal{M}})^{\perp})$. Finally, for every $X \in T^{1,0}\tilde{\mathcal{M}}$ and $Y, Y' \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}}))$, we have

$$\begin{aligned} X(\tilde{g}(Y, Y')) &= \tilde{g}(D^*_X Y, Y') + \tilde{g}(Y, D^*_{\overline{X}} Y') \\ &= \tilde{g}(\mathcal{D}_X Y + B(X, Y), Y') + \tilde{g}(Y, \mathcal{D}_{\overline{X}} Y' + B(X, Y')) \\ &= \tilde{g}(\mathcal{D}_X Y, Y') + \tilde{g}(Y, \mathcal{D}_{\overline{X}} Y'), \end{aligned}$$

which proves that $\mathcal{D}$ preserves $g$. ■

We call $B$ the second fundamental form of $\mathcal{V}(\tilde{\mathcal{M}})$. Obviously $B(Y, Y')$ is complex bilinear in both $Y$ and $Y'$, but it is not symmetric in $Y$ and $Y'$ in general.

Similarly for every $X \in T^{1,0}\tilde{\mathcal{M}}$ and $Z \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}})^{\perp})$ we define $\mathcal{D}^{\perp}$ and $A$ by setting

$$D^*_X Z = -A_Z X + \mathcal{D}^{\perp}_X Z. \tag{3.16}$$

**Proposition 3.3.** (1) $\mathcal{D}^{\perp}$ *is the Hermitian connection of* $\mathcal{Q}$*;* (2) $A$ *is a* $(0,1)$*-form with values in* $\mathrm{Hom}(\mathcal{V}(\tilde{\mathcal{M}})^{\perp}, \mathcal{V}(\tilde{\mathcal{M}}))$*;* (3) $\tilde{g}(B(X,Y),Z) = \tilde{g}(Y, A_Z X)$ *for* $Y \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}}))$, $Z \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}})^{\perp})$.

**Proof.** As in (3.15) we can see that $\mathcal{D}^{\perp}$ defines a connection in $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$ which preserves $\tilde{g}|_{\mathcal{V}(\tilde{\mathcal{M}})^{\perp}}$.

Let $\tilde{Y}$ be a local holomorphic section of $\mathcal{Q}$, $Y'$ the corresponding $C^{\infty}$ section of $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$ under the identification $\mathcal{Q} = \mathcal{V}(\tilde{\mathcal{M}})^{\perp}$, and $Z$ a local holomorphic section of $\mathcal{V}(\tilde{M}^*)$ representing $\tilde{Y}$. Let

$$Z = Y + Y' \quad \text{with } Y \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}})).$$

Then

$$\begin{aligned} D^*_X Z &= D^*_X Y + D^*_X Y' = \mathcal{D}_X Y + B(X,Y) - A_{Y'}X + \mathcal{D}^{\perp}_X Y' \\ &= (\mathcal{D}_X Y - A_{Y'}X) + (B(X,Y) + \mathcal{D}^{\perp}_X Y'). \end{aligned}$$

Since $D^*Z$ is a $(1,0)$-form with values in $\mathcal{V}(\tilde{M}^*)$, $\mathcal{D}Y - A_{Y'}$ and $B(\cdot,Y) + \mathcal{D}^{\perp}Y'$ are also $(1,0)$-forms with values in $\mathcal{V}(\tilde{\mathcal{M}})$ and $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$, respectively. Since $B(\cdot,Y)$ is a $(1,0)$-form by (3.15), it follows that $\mathcal{D}^{\perp}Y'$ is a $(1,0)$-form. This shows that the corresponding connection $\mathcal{D}^{\perp}$ is the Hermitian connection of $\mathcal{Q}$.

Finally, if $Y \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}}))$, $Z \in \mathcal{X}(\mathcal{V}(\tilde{\mathcal{M}})^{\perp})$, then

$$\begin{aligned} 0 &= X\tilde{g}(Y,Z) = \tilde{g}(D^*_X Y, Z) + \tilde{g}(Y, D^*_X Z) \\ &= \tilde{g}(\mathcal{D}_X Y + B(X,Y), Z) + \tilde{g}(Y, -A_Z X + \mathcal{D}^{\perp}_X Z) \\ &= \tilde{g}(B(X,Y),Z) - \tilde{g}(Y, A_Z X). \end{aligned}$$

This shows that $A$ is a $(0,1)$-form since $B$ is a $(1,0)$-form. ■

Similar to the Chern–Finsler connection $D$ in $\mathcal{V}(\tilde{M})$, using the almost product structure $Q$ and the $h(v)$ projections of $T^{1,0}\tilde{\mathcal{M}}$ one can extend the connection $\mathcal{D}$ and the second fundamental form $B$ to the whole $T^{1,0}\tilde{\mathcal{M}}$, i.e., for every $X \in T^{1,0}\tilde{\mathcal{M}}$ and $Y \in \mathcal{X}(T^{1,0}\tilde{\mathcal{M}})$, we define

$$\mathcal{D}_X Y = \mathcal{D}_X vY + Q[\mathcal{D}_X QhY],\ B(X,Y) = B(X, vY) + QB(X, QhY).$$

The extended connection, also denoted by $\mathcal{D}$, is obviously a Hermitian connection in $T^{1,0}\tilde{\mathcal{M}}$. The extended fundamental form, also denoted by $B$,is called the second fundamental form of the complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$.

Now for a local holomorphic frame $\{\frac{\partial}{\partial \eta^i}\}$ of $\mathcal{V}(\tilde{\mathcal{M}})$ we may put

$$D^*_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial \eta^j} = \mathcal{D}_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial \eta^j} + B\left(\frac{\delta}{\delta w^k}, \frac{\partial}{\partial \eta^j}\right), \tag{3.17}$$

$$D^*_{\frac{\delta}{\delta \eta^k}} \frac{\partial}{\partial \eta^j} = \mathcal{D}_{\frac{\partial}{\partial \eta^k}} \frac{\partial}{\partial \eta^j} + B\left(\frac{\partial}{\partial \eta^k}, \frac{\partial}{\partial \eta^j}\right), \tag{3.18}$$

and denote by

$$\mathcal{D}_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial \eta^j} = \mathfrak{F}^i_{j;k} \frac{\partial}{\partial \eta^i},\ B\left(\frac{\delta}{\delta w^k}, \frac{\partial}{\partial \eta^j}\right) = \mathscr{B}^a_{j;k} B_a,$$

$$\mathcal{D}_{\frac{\partial}{\partial \eta^k}} \frac{\partial}{\partial \eta^j} = \mathfrak{F}^i_{jk} \frac{\partial}{\partial \eta^i},\ B\left(\frac{\partial}{\partial \eta^k}, \frac{\partial}{\partial \eta^j}\right) = \mathscr{B}^a_{jk} B_a.$$

The extended Hermitian connection $\mathcal{D}$ is also called the induced complex Finsler connection on $(\mathcal{M}, \mathcal{F})$. We denote by $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$ the induced complex Finsler connection on $(\mathcal{M}, \mathcal{F})$. The tensor fields $\mathscr{B}^a_{j;k}$ is called the horizontal components of the second fundamental form $B$ and $\mathscr{B}^a_{jk}$ is called the vertical components of the second fundamental form $B$. Formulas of (3.17) and (3.18) are called Gauss's formulas for the holomorphic immersion of $(\mathcal{M}, \mathcal{F})$ into $(M, F)$.

**Theorem 3.4.** *Let* $(M, F)$ *be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection* $(N^{\alpha}_{\mu}, F^{\alpha}_{\beta;\mu}, F^{\alpha}_{\beta\mu})$ *and* $(\mathcal{M}, \mathcal{F})$ *be the complex Finsler submanifold of* $(M, F)$*. Then we have the following assertions:*

(1) *The local coefficients of the induced complex linear connection* $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$ *are given by* (3.2) *and respectively by*

$$\mathfrak{F}^i_{j;k} = \mathcal{B}^i_\gamma (B^\gamma_{jk} + B^\alpha_j F^{*\gamma}_{\alpha;k}), \tag{3.19}$$

$$\mathfrak{F}^i_{jk} = \mathcal{B}^i_\gamma B^\alpha_j F^{*\gamma}_{\alpha k}. \tag{3.20}$$

(2) *The horizontal components* $\mathscr{B}^a_{j;k}$ *and the vertical components* $\mathscr{B}^a_{jk}$ *of the second fundamental form B of* $(\mathcal{M}, \mathcal{F})$ *are given respectively by*

$$\mathscr{B}^a_{j;k} = \mathcal{B}^a_\gamma (B^\gamma_{jk} + B^\alpha_j F^{*\gamma}_{\alpha;k}), \tag{3.21}$$

$$\mathscr{B}^a_{jk} = \mathcal{B}^a_\gamma B^\alpha_j F^{*\gamma}_{\alpha k}. \tag{3.22}$$

**Proof.** By (3.10) and (3.4) we have

$$\begin{aligned} D^*_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial \eta^j} &= D^*_{\frac{\delta}{\delta w^k}} \left( B^\alpha_j \frac{\partial}{\partial v^\alpha} \right) = (B^\gamma_{jk} + B^\alpha_j F^{*\gamma}_{\alpha;k}) \frac{\partial}{\partial v^\gamma} \\ &= \mathcal{B}^i_\gamma (B^\gamma_{jk} + B^\alpha_j F^{*\gamma}_{\alpha;k}) \frac{\partial}{\partial \eta^i} + \mathcal{B}^a_\gamma (B^\gamma_{jk} + B^\alpha_j F^{*\gamma}_{\alpha;k}) B_a. \end{aligned} \tag{3.23}$$

Comparing (3.23) with (3.17) we obtain (3.19) and (3.21). Similarly, using (3.4) and (3.10) we have

$$\begin{aligned} D^*_{\frac{\partial}{\partial \eta^k}} \frac{\partial}{\partial \eta^j} &= D^*_{\frac{\partial}{\partial \eta^k}} \left( B^\alpha_j \frac{\partial}{\partial v^\alpha} \right) = B^\alpha_j F^{*\gamma}_{\alpha k} \frac{\partial}{\partial v^\gamma} \\ &= \mathcal{B}^i_\gamma B^\alpha_j F^{*\gamma}_{\alpha k} \frac{\partial}{\partial \eta^i} + \mathcal{B}^a_\gamma B^\alpha_j F^{*\gamma}_{\alpha k} B_a. \end{aligned} \tag{3.24}$$

Comparing (3.24) with (3.18) we get (3.20) and (3.22). This completes the proof. ■

The main distinction between the induced Finsler connection on a real submanifold of a real Finsler manifold and the induced complex Finsler connection on a complex submanifold of a complex Finsler manifold is that, in general the former may not necessary coincide with the intrinsic Finsler connection of the real Finsler submanifold while the latter is exactly the same as the intrinsic complex Finsler connection of the complex Finsler submanifold. This interesting fact was proved in G. Munteanu [8, p. 136, Theorem 5.4.4] for the more general Chern–Lagrange spaces, especially it is true for a strongly pseudoconvex complex Finsler manifold endowed with the Chern–Finsler connection. The distinction between the real and complex Finsler submanifolds shows that the geometry of the complex Finsler submanifolds sharing more analogous to the geometry of the Riemannian submanifolds. In the following we treat the induced Finsler connection *IFC* and the intrinsic Chern–Finsler connection on a complex Finsler submanifold as the same object and call $\mathcal{D}$ the Chern–Finsler connection of $(\mathcal{M}, \mathcal{F})$.

Now we derive the Hermitian connection $\mathcal{D}^\perp$ in $\mathcal{V}(\tilde{\mathcal{M}})^\perp$ in local coordinates. We denote $A_a := A_{B_a}$. Since the local frame $\{B_a\}$ for $\mathcal{Q} = \mathcal{V}(\tilde{\mathcal{M}})^\perp$ (by identification) is only $C^\infty$, using (3.16) we shall put

$$D^*_{\frac{\delta}{\delta w^k}} B_a = -A_a \frac{\delta}{\delta w^k} + \mathcal{D}^\perp_{\frac{\delta}{\delta w^k}} B_a, \tag{3.25}$$

$$D^*_{\frac{\delta}{\delta \bar{w}^k}} B_a = -A_a \frac{\delta}{\delta \bar{w}^k} + \mathcal{D}^\perp_{\frac{\delta}{\delta \bar{w}^k}} B_a, \tag{3.26}$$

$$D^*_{\frac{\partial}{\partial \eta^k}} B_a = -A_a \frac{\partial}{\partial \eta^k} + \mathcal{D}^\perp_{\frac{\partial}{\partial \eta^k}} B_a, \tag{3.27}$$

$$D^*_{\frac{\partial}{\partial \bar{\eta}^k}} B_a = -A_a \frac{\partial}{\partial \bar{\eta}^k} + \mathcal{D}^\perp_{\frac{\partial}{\partial \bar{\eta}^k}} B_a, \tag{3.28}$$

where we set

$$A_a \frac{\delta}{\delta w^k} = \mathscr{A}^i_{a;k} \frac{\partial}{\partial \eta^i}, \qquad \mathcal{D}^\perp_{\frac{\delta}{\delta w^k}} B_a = \mathscr{F}^b_{a;k} B_b, \tag{3.29}$$

$$A_a \frac{\delta}{\delta \bar{w}^k} = \mathscr{A}^i_{a;\bar{k}} \frac{\partial}{\partial \eta^i}, \qquad \mathcal{D}^{\perp}_{\frac{\delta}{\delta \bar{w}^k}} B_a = \mathscr{F}^b_{a;\bar{k}} B_b, \tag{3.30}$$

$$A_a \frac{\partial}{\partial \eta^k} = \mathscr{A}^i_{ak} \frac{\partial}{\partial \eta^i}, \qquad \mathcal{D}^{\perp}_{\frac{\partial}{\partial \eta^k}} B_a = \mathscr{F}^b_{ak} B_b, \tag{3.31}$$

$$A_a \frac{\partial}{\partial \bar{\eta}^k} = \mathscr{A}^i_{a\bar{k}} \frac{\partial}{\partial \eta^i}, \qquad \mathcal{D}^{\perp}_{\frac{\partial}{\partial \bar{\eta}^k}} B_a = \mathscr{F}^b_{a\bar{k}} B_b. \tag{3.32}$$

$\mathcal{D}^{\perp}$ is also called the complex normal Finsler connection in $\mathcal{V}(\tilde{\mathcal{M}})^{\perp}$. The formulas (3.25)–(3.28) are called Weingarten's formulas.

**Theorem 3.5.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection $(N^{\alpha}_{\mu}, F^{\alpha}_{\beta;\mu}, F^{\alpha}_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$. Then we have the following assertions:*

(1) *The local coefficients $(\mathcal{N}^i_j, \mathscr{F}^b_{a;k}, \mathscr{F}^b_{a;\bar{k}}, \mathscr{F}^b_{ak}, \mathscr{F}^b_{a\bar{k}})$ of the NFC are given by (3.2) and respectively by*

$$\mathscr{F}^b_{a;k} = \mathcal{B}^b_{\gamma} \left[ \frac{\delta}{\delta w^k}(B^{\gamma}_a) + B^{\alpha}_a F^{*\gamma}_{\alpha;k} \right], \qquad \mathscr{F}^b_{a;\bar{k}} = \mathcal{B}^b_{\gamma} \frac{\delta}{\delta \bar{w}^k}(B^{\gamma}_a), \tag{3.33}$$

$$\mathscr{F}^b_{ak} = \mathcal{B}^b_{\gamma} \left[ \frac{\partial}{\partial \eta^k}(B^{\gamma}_a) + B^{\alpha}_a F^{*\gamma}_{\alpha k} \right], \qquad \mathscr{F}^b_{a\bar{k}} = \mathcal{B}^b_{\gamma} \frac{\partial}{\partial \bar{\eta}^k}(B^{\gamma}_a). \tag{3.34}$$

(2) *The local coefficients $\mathscr{A}^i_{a;k}, \mathscr{A}^i_{a;\bar{k}}, \mathscr{A}^i_{ak}, \mathscr{A}^i_{a\bar{k}}$ are given respectively by*

$$\mathscr{A}^i_{a;k} = -\mathcal{B}^i_{\gamma} \left[ \frac{\delta}{\delta w^k}(B^{\gamma}_a) + B^{\alpha}_a F^{*\gamma}_{\alpha;k} \right], \qquad \mathscr{A}^i_{a;\bar{k}} = -\mathcal{B}^i_{\gamma} \frac{\delta}{\delta \bar{w}^k}(B^{\gamma}_a), \tag{3.35}$$

$$\mathscr{A}^i_{ak} = -\mathcal{B}^i_{\gamma} \left[ \frac{\partial}{\partial \eta^k}(B^{\gamma}_a) + B^{\alpha}_a F^{*\gamma}_{\alpha k} \right], \qquad \mathscr{A}^i_{a\bar{k}} = -\mathcal{B}^i_{\gamma} \frac{\partial}{\partial \bar{\eta}^k}(B^{\gamma}_a). \tag{3.36}$$

**Proof.** Since $B_a = B^{\gamma}_a \frac{\partial}{\partial v^{\gamma}}$, using (3.10) and (3.11), we have

$$D^*_{\frac{\delta}{\delta w^k}} B_a = D_{\frac{\delta}{\delta w^k}} \left( B^{\alpha}_a \frac{\partial}{\partial v^{\alpha}} \right) = \left[ \frac{\delta}{\delta w^k}(B^{\gamma}_a) + B^{\alpha}_a F^{*\gamma}_{\alpha;k} \right] \frac{\partial}{\partial v^{\gamma}},$$
$$D^*_{\frac{\delta}{\delta \bar{w}^k}} B_a = D_{\frac{\delta}{\delta \bar{w}^k}} \left( B^{\alpha}_a \frac{\partial}{\partial v^{\alpha}} \right) = \frac{\delta}{\delta \bar{w}^k}(B^{\gamma}_a) \frac{\partial}{\partial v^{\gamma}},$$
$$D^*_{\frac{\partial}{\partial \eta^k}} B_a = D_{\frac{\partial}{\partial \eta^k}} \left( B^{\alpha}_a \frac{\partial}{\partial v^{\alpha}} \right) = \left[ \frac{\partial}{\partial \eta^k}(B^{\gamma}_a) + B^{\alpha}_a F^{*\gamma}_{\alpha k} \right] \frac{\partial}{\partial v^{\gamma}},$$
$$D^*_{\frac{\partial}{\partial \bar{\eta}^k}} B_a = D_{\frac{\partial}{\partial \bar{\eta}^k}} \left( B^{\alpha}_a \frac{\partial}{\partial v^{\alpha}} \right) = \frac{\partial}{\partial \bar{\eta}^k}(B^{\gamma}_a) \frac{\partial}{\partial v^{\gamma}}.$$

By substituting (3.4) into the above equations and comparing the corresponding coefficients with (3.25)–(3.28), we obtain (3.53) and (3.52), this completes the proof. ■

**Proposition 3.6.** *The $h(v)$ components of the second fundamental form $B$ of $(\mathcal{M}, \mathcal{F})$ and the normal connection coefficients $\mathscr{A}^l_{a;\bar{k}}, \mathscr{A}^l_{a\bar{k}}$ of $\mathcal{D}^{\perp}$ satisfy the following equalities*

$$\mathscr{B}^a_{j;k} = g_{j\bar{l}} \overline{\mathscr{A}^l_{a;\bar{k}}}, \qquad \mathscr{B}^a_{jk} = g_{j\bar{l}} \overline{\mathscr{A}^l_{a\bar{k}}}. \tag{3.37}$$

**Proof.** Since $\tilde{g}(\frac{\partial}{\partial \eta^j}, B_a) = 0$, differentiating covariantly with respect to $\frac{\delta}{\delta w^k}$ for the Hermitian connection $D$ we have

$$0 = \tilde{g} \left( D_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial \eta^j}, B_a \right) + \tilde{g} \left( \frac{\partial}{\partial \eta^j}, D_{\frac{\delta}{\delta \bar{w}^k}} B_a \right).$$

So that

$$\tilde{g}\left(\mathcal{D}_{\frac{\delta}{\delta w^k}}\frac{\partial}{\partial \eta^j}+B\left(\frac{\delta}{\delta w^k},\frac{\partial}{\partial \eta^j}\right),B_a\right)+\tilde{g}\left(\frac{\partial}{\partial \eta^j},-A_a\frac{\delta}{\delta \bar{w}^k}+\mathcal{D}^{\perp}_{\frac{\delta}{\delta \bar{w}^k}}B_a\right)=0.$$

Since

$$\tilde{g}\left(\mathcal{D}_{\frac{\delta}{\delta w^k}}\frac{\partial}{\partial \eta^j},B_a\right)=\tilde{g}\left(\frac{\partial}{\partial \eta^j},\mathcal{D}^{\perp}_{\frac{\delta}{\delta \bar{w}^k}}B_a\right)=0.$$

We have

$$\tilde{g}\left(B\left(\frac{\delta}{\delta w^k},\frac{\partial}{\partial \eta^j}\right),B_a\right)=\tilde{g}\left(\frac{\partial}{\partial \eta^j},A_a\frac{\delta}{\delta \bar{w}^k}\right)=\tilde{g}\left(\frac{\partial}{\partial \eta^j},\mathscr{A}^l_{a;\bar{k}}\frac{\partial}{\partial \eta^l}\right),$$

i.e.,

$$\mathscr{B}^a_{j;k}=g_{j\bar{l}}\overline{\mathscr{A}^l_{a;\bar{k}}}.$$

Similarly we can prove the second equality. This completes the proof. ■

Note that locally $B^\gamma_i$ transform as

$$B^\gamma_i=B^{\alpha'}_{j'}\frac{\partial w^{j'}}{\partial w^i}\frac{\partial z^\gamma}{\partial z^{\alpha'}},\qquad \mathcal{B}^i_\gamma=\mathcal{B}^{j'}_{\beta'}\frac{\partial w^i}{\partial w^{j'}}\frac{\partial z^{\beta'}}{\partial z^\gamma},$$

they are the components of a mixed complex Finsler tensor field on $(\mathcal{M},\mathcal{F})$ with respect to the holomorphic vector bundle $\mathcal{V}(\tilde{M}^*)$ endowed with the connection pair $(IVFC, IFC)$. Therefore we can define the horizontal and vertical relative covariant derivatives of $B^\gamma_i$ with respect to the connection pair $(IVFC, IFC)$, for example,

$$B^\gamma_{i|j}=\frac{\delta}{\delta w^j}(B^\gamma_i)+B^\beta_i F^{*\gamma}_{\beta;j}-B^\gamma_k\mathfrak{F}^k_{i;j}=B^\gamma_{ij}+B^\beta_i F^{*\gamma}_{\beta;j}-B^\gamma_k\mathfrak{F}^k_{i;j}, \tag{3.38}$$

$$B^\gamma_{i|\bar{j}}=\frac{\delta}{\delta \bar{w}^j}(B^\gamma_i)+B^\beta_i F^{*\gamma}_{\beta;\bar{j}}-B^\gamma_k\mathfrak{F}^k_{i;\bar{j}}=0, \tag{3.39}$$

$$B^\gamma_{i\|j}=\frac{\partial}{\partial \eta^j}(B^\gamma_i)+B^\beta_i F^{*\gamma}_{\beta j}-B^\gamma_k\mathfrak{F}^k_{ij}=B^\beta_i F^{*\gamma}_{\beta j}-B^\gamma_k\mathfrak{F}^k_{ij}, \tag{3.40}$$

$$B^\gamma_{i\|\bar{j}}=\frac{\partial}{\partial \bar{\eta}^j}(B^\gamma_i)+B^\beta_i F^{*\gamma}_{\beta\bar{j}}-B^\gamma_k\mathfrak{F}^k_{i\bar{j}}=0. \tag{3.41}$$

Since locally $H^a_i B^\gamma_a$ transform as

$$H^a_i B^\gamma_a=H^{c'}_{j'}B^{\beta'}_{c'}\frac{\partial w^{j'}}{\partial w^i}\frac{\partial z^\gamma}{\partial z^{\beta'}},$$

they are the components of a mixed complex Finsler tensor field on $(\mathcal{M},\mathcal{F})$ with respect to the holomorphic vector bundle $\mathcal{V}(\tilde{M}^*)$ endowed with the connection pair $(IVFC, IFC)$. Therefore the horizontal and vertical relative covariant derivatives of $H^a_i B^\gamma_a$ with respect to the connection pair $(IVFC, IFC)$ are given by

$$(H^a_i B^\gamma_a)_{|j}=\frac{\delta}{\delta w^j}(H^a_i B^\gamma_a)+(H^a_i B^\beta_a)F^{*\gamma}_{\beta;j}-(H^a_k B^\gamma_a)\mathfrak{F}^k_{i;j},$$

$$(H^a_i B^\gamma_a)_{|\bar{j}}=\frac{\delta}{\delta \bar{w}^j}(H^a_i B^\gamma_a)+(H^a_i B^\beta_a)F^{*\gamma}_{\beta\bar{j}}-(H^a_k B^\gamma_a)\mathfrak{F}^k_{i;\bar{j}},$$

$$(H^a_i B^\gamma_a)_{\|j}=\frac{\partial}{\partial \eta^j}(H^a_i B^\gamma_a)+(H^a_i B^\beta_a)F^{*\gamma}_{\beta j}-(H^a_k B^\gamma_a)\mathfrak{F}^k_{ij},$$

$$(H^a_i B^\gamma_a)_{\|\bar{j}}=\frac{\partial}{\partial \bar{\eta}^j}(H^a_i B^\gamma_a)+(H^a_i B^\beta_a)F^{*\gamma}_{\beta\bar{j}}-(H^a_k B^\gamma_a)\mathfrak{F}^k_{i\bar{j}}.$$

If we define

$$B^{\gamma}_{a|j} = \frac{\delta}{\delta w^j}(B^{\gamma}_a) + B^{\beta}_a F^{*\gamma}_{\beta;j} - B^{\gamma}_b \mathscr{F}^b_{a;j}, \tag{3.42}$$

$$B^{\gamma}_{a|\bar{j}} = \frac{\delta}{\delta \bar{w}^j}(B^{\gamma}_a) - B^{\gamma}_b \mathscr{F}^b_{a;\bar{j}}, \tag{3.43}$$

$$B^{\gamma}_{a\|j} = \frac{\partial}{\partial \eta^j}(B^{\gamma}_a) + B^{\beta}_a F^{*\gamma}_{\beta j} - B^{\gamma}_b \mathscr{F}^b_{aj}, \tag{3.44}$$

$$B^{\gamma}_{a\|\bar{j}} = \frac{\partial}{\partial \bar{\eta}^j}(B^{\gamma}_a) - B^{\gamma}_b \mathscr{F}^b_{a\bar{j}} \tag{3.45}$$

the $h(v)$-relative covariant derivative of the mixed tensor field $B^{\gamma}_a$ with respect to the connection pair $(IVFC, NFC)$ and

$$H^a_{i|j} = \frac{\delta}{\delta w^j}(H^a_i) + H^b_i \mathscr{F}^a_{b;j} - H^a_k \mathfrak{F}^k_{i;j}, \tag{3.46}$$

$$H^a_{i|\bar{j}} = \frac{\delta}{\delta \bar{w}^j}(H^a_i) + H^b_i \mathscr{F}^a_{b;\bar{j}}, \tag{3.47}$$

$$H^a_{i\|j} = \frac{\partial}{\partial \eta^j}(H^a_i) + H^b_i \mathscr{F}^a_{bj} - H^a_k \mathfrak{F}^k_{ij}, \tag{3.48}$$

$$H^a_{i\|\bar{j}} = \frac{\partial}{\partial \bar{\eta}^j}(H^a_i) + H^b_i \mathscr{F}^a_{b\bar{j}} \tag{3.49}$$

the $h(v)$-relative covariant derivative of the mixed tensor field $H^a_i$ with respect to the connection pair $(IFC, NFC)$. Then we have

$$(H^a_i B^{\gamma}_a)_{|j} = H^a_{i|j} B^{\gamma}_a + H^a_i B^{\gamma}_{a|j}, \qquad (H^a_i B^{\gamma}_a)_{|\bar{j}} = H^a_{i|\bar{j}} B^{\gamma}_a + H^a_i B^{\gamma}_{a|\bar{j}},$$
$$(H^a_i B^{\gamma}_a)_{\|j} = H^a_{i\|j} B^{\gamma}_a + H^a_i B^{\gamma}_{a\|j}, \qquad (H^a_i B^{\gamma}_a)_{\|\bar{j}} = H^a_{i\|\bar{j}} B^{\gamma}_a + H^a_i B^{\gamma}_{a\|\bar{j}}.$$

**Proposition 3.7.** *The $h(v)$-relative covariant derivative of $B^{\gamma}_a$ satisfies*

$$B^{\gamma}_{i|j} = B^{\gamma}_a \mathscr{B}^a_{i;j}, \qquad B^{\gamma}_{i\|j} = B^{\gamma}_a \mathscr{B}^a_{ij}, \tag{3.50}$$

$$\mathscr{A}^i_{a;k} = -\mathcal{B}^i_{\gamma} B^{\gamma}_{a|k}, \qquad \mathscr{A}^i_{a;\bar{k}} = -\mathcal{B}^i_{\gamma} B^{\gamma}_{a|\bar{k}}, \tag{3.51}$$

$$\mathscr{A}^i_{ak} = -\mathcal{B}^i_{\gamma} B^{\gamma}_{a\|k}, \qquad \mathscr{A}^i_{a\bar{k}} = -\mathcal{B}^i_{\gamma} B^{\gamma}_{a\|\bar{k}}, \tag{3.52}$$

$$\mathscr{F}^b_{a;k} = \mathcal{B}^b_{\gamma} B^{\gamma}_{a|k}, \qquad \mathscr{F}^b_{a;\bar{k}} = \mathcal{B}^b_{\gamma} B^{\gamma}_{a|\bar{k}}, \tag{3.53}$$

$$\mathscr{F}^b_{ak} = \mathcal{B}^b_{\gamma} B^{\gamma}_{a\|k}, \qquad \mathscr{F}^b_{a\bar{k}} = \mathcal{B}^b_{\gamma} B^{\gamma}_{a\|\bar{k}}. \tag{3.54}$$

**Proof.** We only prove the first identity. Using (3.19), (1.15) and (3.21), we have

$$\begin{aligned} B^{\beta}_a \mathscr{B}^a_{j;k} &= (\delta^{\beta}_{\gamma} - B^{\beta}_i \mathcal{B}^i_{\gamma})(B^{\gamma}_{jk} + B^{\alpha}_j F^{*\gamma}_{\alpha;k}) \\ &= B^{\beta}_{jk} + B^{\alpha}_j F^{*\beta}_{\alpha;k} - B^{\beta}_i \mathcal{B}^i_{\gamma}(B^{\gamma}_{jk} + B^{\alpha}_j F^{*\gamma}_{\alpha;k}) \\ &= B^{\beta}_{jk} + B^{\alpha}_j F^{*\beta}_{\alpha;k} - B^{\beta}_i \mathfrak{F}^i_{j;k} \\ &= B^{\beta}_{j|k}. \end{aligned}$$

The other identities are similar to obtain. ■

**Proposition 3.8.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection $CFC = (N^{\alpha}_{\mu}, F^{\alpha}_{\beta;\mu}, F^{\alpha}_{\beta\mu})$. Let $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(\mathcal{M}, \mathcal{F})$ endowed with the induced Finsler* *connection IFC* $= (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$. *Then their corresponding* $h(v)$*-covariant derivatives are related by*

$$g_{i\bar{j}|k} = B^\mu_i B^{\bar{\nu}}_{\bar{j}}(B^\alpha_k \tilde{g}_{\mu\bar{\nu}|\alpha} + H^a_k B^\alpha_a \tilde{g}_{\mu\bar{\nu}\|\alpha}). \tag{3.55}$$

$$g_{i\bar{j}\|k} = B^\alpha_k B^\mu_i B^{\bar{\nu}}_{\bar{j}} \tilde{g}_{\mu\bar{\nu}\|\alpha}. \tag{3.56}$$

**Proof.** It is a direct calculation. In fact, using (1.14)–(1.16), (2.1) and (3.5), we have

$$\begin{aligned} g_{i\bar{j}|k} &= \frac{\delta}{\delta w^k}(g_{i\bar{j}}) - g_{l\bar{j}}\mathfrak{F}^l_{i;k} \\ &= B^\alpha_k \frac{\delta}{\delta z^\alpha}(\tilde{g}_{\mu\bar{\nu}} B^\mu_i B^{\bar{\nu}}_{\bar{j}}) + H^a_k B^\alpha_a \frac{\partial}{\partial v^\alpha}(\tilde{g}_{\mu\bar{\nu}} B^\mu_i B^{\bar{\nu}}_{\bar{j}}) - \tilde{g}_{\mu\bar{\nu}} B^\mu_l B^{\bar{\nu}}_{\bar{j}} \mathcal{B}^l_\alpha (B^\alpha_{ik} + B^\gamma_i B^\sigma_k F^\alpha_{\gamma;\sigma} + B^\gamma_i H^a_k B^\beta_a F^\alpha_{\gamma\beta}) \\ &= B^\alpha_k B^\mu_i B^{\bar{\nu}}_{\bar{j}} \left[\frac{\delta}{\delta z^\alpha}(\tilde{g}_{\mu\bar{\nu}}) - \tilde{g}_{\beta\bar{\nu}} F^\beta_{\mu;\alpha}\right] + H^a_k B^\alpha_a B^\mu_i B^{\bar{\nu}}_{\bar{j}} \left[\frac{\partial}{\partial v^\alpha}(\tilde{g}_{\mu\bar{\nu}}) - \tilde{g}_{\beta\bar{\nu}} F^\beta_{\mu\alpha}\right] \\ &= B^\mu_i B^{\bar{\nu}}_{\bar{j}}(B^\alpha_k \tilde{g}_{\mu\bar{\nu}|\alpha} + H^a_k B^\alpha_a \tilde{g}_{\mu\bar{\nu}\|\alpha}), \end{aligned}$$

and

$$\begin{aligned} g_{i\bar{j}\|k} &= B^\alpha_k \frac{\partial}{\partial v^\alpha}(\tilde{g}_{\mu\bar{\nu}} B^\mu_i B^{\bar{\nu}}_{\bar{j}}) - g_{l\bar{j}}\mathfrak{F}^l_{ik} \\ &= B^\alpha_k B^\mu_i B^{\bar{\nu}}_{\bar{j}} \frac{\partial}{\partial v^\alpha}(\tilde{g}_{\mu\bar{\nu}}) - \tilde{g}_{\mu\bar{\nu}} B^\mu_l B^{\bar{\nu}}_{\bar{j}} \mathcal{B}^l_\alpha B^\beta_i B^\gamma_k F^\alpha_{\beta\gamma} \\ &= B^\alpha_k B^\mu_i B^{\bar{\nu}}_{\bar{j}} \frac{\partial}{\partial v^\alpha}(\tilde{g}_{\mu\bar{\nu}}) - \tilde{g}_{\mu\bar{\nu}} B^{\bar{\nu}}_{\bar{j}} (\delta^\mu_\alpha - B^\mu_a \mathcal{B}^a_\alpha) B^\beta_i B^\gamma_k F^\alpha_{\beta\gamma} \\ &= B^\alpha_k B^\mu_i B^{\bar{\nu}}_{\bar{j}} \left[\frac{\partial}{\partial v^\alpha}(\tilde{g}_{\mu\bar{\nu}}) - \tilde{g}_{\beta\bar{\nu}} F^\beta_{\mu\alpha}\right] \\ &= B^\alpha_k B^\mu_i B^{\bar{\nu}}_{\bar{j}} \tilde{g}_{\mu\bar{\nu}\|\alpha}, \end{aligned}$$

where we have denoted $\tilde{g}_{\mu\bar{\nu}|\alpha}$ and $\tilde{g}_{\mu\bar{\nu}\|\alpha}$ the $h(v)$-covariant derivatives of $\tilde{g}_{\mu\bar{\nu}}$ with respect to the Chern–Finsler connection *CFC* on $(M, F)$. ■

It also follows from the above equations that

$$g_{i\bar{j}|k} = 0, \qquad g_{i\bar{j}\|k} = 0, \tag{3.57}$$

since the Chern–Finsler connection on $(M, F)$ is both $h$-metrical and $v$-metrical.

## 4. Local expression of Gauss–Codazzi–Ricci equations

Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold endowed with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$, and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$. In the previous section we obtain three complex linear connections

$$IVFC = (\mathcal{N}^i_j, F^{*\gamma}_{\beta;k}, F^{*\gamma}_{\beta k}), \qquad IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk}), \qquad NFC = (\mathcal{N}^i_j, \mathscr{F}^b_{a;k}, \mathscr{F}^b_{a;\bar{k}}, \mathscr{F}^b_{ak}, \mathscr{F}^b_{a\bar{k}})$$

on the holomorphic vector bundles $\mathcal{V}(\tilde{M}^*)$, $\mathcal{V}(\tilde{\mathcal{M}})$ and respectively the smooth complex vector bundle $\mathcal{V}(\tilde{\mathcal{M}})^\perp$. Hence we have $h(v)$-relative covariant derivative with respect to the connection pairs (*IFC*, *IVFC*) and (*IFC*, *NFC*), respectively. In this section we use these $h(v)$-relative covariant derivatives to derive the curvature relationships between the induced connections and the Chern–Finsler connection *CFC* on the ambient manifold $(M, F)$.

First we shall deal with the curvature of the *IVFC*. We denote by $R^*$ the curvature of *IVFC*. Then the curvature form $R^*$ of *IVFC* and the curvature form $\tilde{R}$ of *CFC* are related by

$$R^*(X, Y)Z = \tilde{R}(X, Y)Z, \quad \forall X, Y \in \mathcal{X}(T^{1,0}\tilde{\mathcal{M}}), Z \in \mathcal{X}(\mathcal{V}(\tilde{M}^*)).$$

Since the $(2,0)$-curvature $\tilde{\Omega}$ of *CFC* vanishes identically, using (1.8), (2.18) and (3.5), we have $R^*(X,Y)Z = \tilde{\Omega}(X,Y)Z = 0$ for all $X, Y \in \mathcal{X}(T^{1,0}\tilde{\mathcal{M}})$ and $Z \in \mathcal{X}(\mathcal{V}(\tilde{M}^*))$. Thus we need only deal with the $(1,1)$-curvature of *IVFC*. Locally, if we put

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial v^\gamma} = R^{*\sigma}_{\gamma;i\bar{j}}\frac{\partial}{\partial v^\sigma}, \qquad R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial v^\gamma} = R^{*\sigma}_{\gamma i;\bar{j}}\frac{\partial}{\partial v^\sigma}, \tag{4.1}$$

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial v^\gamma} = R^{*\sigma}_{\gamma\bar{j};i}\frac{\partial}{\partial v^\sigma}, \qquad R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial v^\gamma} = R^{*\sigma}_{\gamma i\bar{j}}\frac{\partial}{\partial v^\sigma}. \tag{4.2}$$

Then using (2.19)–(2.21) and (3.5), we have

$$\begin{aligned}
R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial v^\gamma} &= R^*\left(B^\alpha_i\frac{\delta}{\delta z^\alpha} + H^a_i B_a, B^{\bar{\beta}}_{\bar{j}}\frac{\delta}{\delta \bar{z}^\beta} + H^{\bar{b}}_{\bar{j}} B_{\bar{b}}\right)\frac{\partial}{\partial v^\gamma}\\
&= B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{\Omega}\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta \bar{z}^\beta}\right)\frac{\partial}{\partial v^\gamma} + B^\alpha_i H^{\bar{b}}_{\bar{j}}\tilde{\Omega}\left(\frac{\delta}{\delta z^\alpha}, B_{\bar{b}}\right)\frac{\partial}{\partial v^\gamma}\\
&\quad + H^a_i B^{\bar{\beta}}_{\bar{j}}\tilde{\Omega}\left(B_a, \frac{\delta}{\delta \bar{z}^\beta}\right)\frac{\partial}{\partial v^\gamma} + H^a_i H^{\bar{b}}_{\bar{j}}\tilde{\Omega}(B_a, B_{\bar{b}})\frac{\partial}{\partial v^\gamma}\\
&= \left(B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}} + B^\alpha_i H^{\bar{b}}_{\bar{j}} B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\bar{\beta};\alpha} + H^a_i B^{\bar{\beta}}_{\bar{j}} B^\alpha_a \tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}} + H^a_i H^{\bar{b}}_{\bar{j}} B^\alpha_a B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}\right)\frac{\partial}{\partial v^\sigma}.
\end{aligned}$$

Thus we get

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial v^\gamma} = R^{*\sigma}_{\gamma;i\bar{j}}\frac{\partial}{\partial v^\sigma}, \tag{4.3}$$

where we have denoted by

$$R^{*\sigma}_{\gamma;i\bar{j}} = B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}} + B^\alpha_i H^{\bar{b}}_{\bar{j}} B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\bar{\beta};\alpha} + H^a_i B^{\bar{\beta}}_{\bar{j}} B^\alpha_a \tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}} + H^a_i H^{\bar{b}}_{\bar{j}} B^\alpha_a B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}. \tag{4.4}$$

By similar calculations, we obtain

$$R^{*\sigma}_{\gamma i;\bar{j}} = B^\alpha_i B^{\bar{b}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}} + B^\alpha_i H^{\bar{b}}_{\bar{j}} B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}, \tag{4.5}$$

$$R^{*\sigma}_{\gamma\bar{j};i} = B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\bar{\beta};\alpha} + H^a_i B^\alpha_a B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}, \tag{4.6}$$

$$R^{*\sigma}_{\gamma i\bar{j}} = B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}. \tag{4.7}$$

Thus we have

**Proposition 4.1.** *Let $(M,F)$ be a strongly pseudoconvex complex Finsler endowed with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$. Then the curvature tensors of IVFC and the curvature of CFC are related by*

$$R^{*\sigma}_{\gamma;i\bar{j}} = B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}} + B^\alpha_i H^{\bar{b}}_{\bar{j}} B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\bar{\beta};\alpha} + H^a_i B^{\bar{\beta}}_{\bar{j}} B^\alpha_a \tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}} + H^a_i H^{\bar{b}}_{\bar{j}} B^\alpha_a B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}, \tag{4.8}$$

$$R^{*\sigma}_{\gamma i;\bar{j}} = B^\alpha_i B^{\bar{b}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}} + B^\alpha_i H^{\bar{b}}_{\bar{j}} B^{\bar{\beta}}_{\bar{b}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}, \tag{4.9}$$

$$R^{*\sigma}_{\gamma\bar{j};i} = B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\bar{\beta};\alpha} + H^a_i B^\alpha_a B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}, \tag{4.10}$$

$$R^{*\sigma}_{\gamma i\bar{j}} = B^\alpha_i B^{\bar{\beta}}_{\bar{j}}\tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}, \tag{4.11}$$

*where $\tilde{R}^\sigma_{\gamma;\alpha\bar{\beta}}, \tilde{R}^\sigma_{\gamma\alpha;\bar{\beta}}, \tilde{R}^\sigma_{\gamma\bar{\beta};\alpha}, \tilde{R}^\sigma_{\gamma\alpha\bar{\beta}}$ are given respectively by (2.24)–(2.27).* ■

Since the induced Finsler connection *IFC* on $(\mathcal{M}, \mathcal{F})$ coincides with the Chern–Finsler connection *CFC* on $(\mathcal{M}, \mathcal{F})$, similar to the curvature and torsion of the complex linear connection that associated to *CFC* on $(M, F)$

in Section 2, if we denote by $R$ the curvature form and $\mathcal{T}$ the torsion of the complex linear connection that associated to *IFC*. Then the non-vanishing components of the curvature $R$ and the non-vanishing components of the torsion $\mathcal{T}$ are given by

$$R\left(\frac{\delta}{\delta w^i},\frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial \eta^k}=R^l_{k;i\bar{j}}\frac{\partial}{\partial \eta^l},\qquad R\left(\frac{\partial}{\partial \eta^i},\frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial \eta^k}=R^l_{ki;\bar{j}}\frac{\partial}{\partial \eta^l},$$
$$R\left(\frac{\delta}{\delta w^i},\frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial \eta^k}=R^l_{k\bar{j};i}\frac{\partial}{\partial \eta^l},\qquad R\left(\frac{\partial}{\partial \eta^i},\frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial \eta^k}=R^l_{ki\bar{j}}\frac{\partial}{\partial \eta^l},$$
$$\mathcal{T}\left(\frac{\delta}{\delta w^i},\frac{\delta}{\delta w^j}\right)=\mathcal{T}^k_{;ji}\frac{\delta}{\delta w^k},\qquad \mathcal{T}\left(\frac{\partial}{\partial \eta^i},\frac{\delta}{\delta w^j}\right)=\mathcal{T}^k_{i;j}\frac{\delta}{\delta w^k},$$
$$\mathcal{T}\left(\frac{\delta}{\delta w^i},\frac{\delta}{\delta \bar{w}^j}\right)=\dot{\mathcal{T}}^k_{;i\bar{j}}\frac{\partial}{\partial \eta^k}+\dot{\mathcal{T}}^{\bar{k}}_{;i\bar{j}}\frac{\partial}{\partial \bar{\eta}^k},\qquad \mathcal{T}\left(\frac{\delta}{\delta w^i},\frac{\partial}{\partial \bar{\eta}^j}\right)=\dot{\mathcal{T}}^k_{\bar{j};i}\frac{\partial}{\partial \eta^k},$$

where

$$R^l_{k;i\bar{j}}=-\frac{\delta}{\delta \bar{w}^j}(\mathfrak{F}^l_{k;i})-\mathfrak{F}^l_{kh}\frac{\delta}{\delta \bar{w}^j}(\mathcal{N}^h_i),\qquad R^l_{ki;\bar{j}}=-\frac{\delta}{\delta \bar{w}^j}(\mathfrak{F}^l_{ki}),\tag{4.12}$$

$$R^l_{k\bar{j};i}=-\frac{\partial}{\partial \bar{\eta}^j}(\mathfrak{F}^l_{k;i})-\mathfrak{F}^l_{kh}\frac{\partial}{\partial \bar{\eta}^j}(\mathcal{N}^h_i),\qquad R^l_{ki\bar{j}}=-\frac{\partial}{\partial \bar{\eta}^j}(\mathfrak{F}^l_{ki}),\tag{4.13}$$

$$\mathcal{T}^k_{;ji}=\mathfrak{F}^k_{j;i}-\mathfrak{F}^k_{i;j},\qquad \mathcal{T}^k_{i;j}=\mathfrak{F}^k_{ji},\qquad \dot{\mathcal{T}}^k_{;i\bar{j}}=-\frac{\delta}{\delta \bar{w}^j}(\mathcal{N}^k_i),\tag{4.14}$$

$$\dot{\mathcal{T}}^{\bar{k}}_{;i\bar{j}}=\frac{\delta}{\delta w^i}(\mathcal{N}^{\bar{k}}_{\bar{j}}),\qquad \dot{\mathcal{T}}^k_{\bar{j};i}=-\frac{\partial}{\partial \bar{\eta}^j}(\mathcal{N}^k_i).\tag{4.15}$$

Using Gauss formula (3.17), (3.18) and Weingarten formula (3.25), (3.27) for $(\mathcal{M},\mathcal{F})$ we derive

$$\begin{aligned}R^*\left(\frac{\delta}{\delta w^i},\frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k}&=R\left(\frac{\delta}{\delta w^i},\frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k}\\&\quad+(\mathscr{B}^a_{k;i}\mathscr{A}^l_{a;j}-\mathscr{B}^a_{k;j}\mathscr{A}^l_{a;i})\frac{\partial}{\partial \eta^l}+(\mathscr{B}^a_{k;j|i}-\mathscr{B}^a_{k;i|j}+\mathcal{T}^l_{;ji}\mathscr{B}^a_{k;l})B_a,\\
R^*\left(\frac{\partial}{\partial \eta^i},\frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k}&=R\left(\frac{\partial}{\partial \eta^i},\frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k}\\&\quad+(\mathscr{B}^a_{ki}\mathscr{A}^l_{a;j}-\mathscr{B}^a_{k;j}\mathscr{A}^l_{ai})\frac{\partial}{\partial \eta^l}+(\mathscr{B}^a_{k;j\|i}-\mathscr{B}^a_{ki|j}+\mathcal{T}^l_{i;j}\mathscr{B}^a_{k;l})B_a,\end{aligned}$$
$$R^*\left(\frac{\partial}{\partial \eta^i},\frac{\partial}{\partial \eta^j}\right)\frac{\partial}{\partial \eta^k}=R\left(\frac{\partial}{\partial \eta^i},\frac{\partial}{\partial \eta^j}\right)\frac{\partial}{\partial \eta^k}+(\mathscr{B}^a_{ki}\mathscr{A}^l_{aj}-\mathscr{B}^a_{kj}\mathscr{A}^l_{ai})\frac{\partial}{\partial \eta^l}+(\mathscr{B}^a_{kj\|i}-\mathscr{B}^a_{ki\|j})B_a,$$

where we have denoted by

$$\mathscr{B}^a_{k;j|i}=\frac{\delta}{\delta w^i}(\mathscr{B}^a_{k;j})+\mathscr{B}^b_{k;j}\mathscr{F}^a_{b;i}-\mathscr{B}^a_{k;l}\mathfrak{F}^l_{j;i}-\mathscr{B}^a_{l;j}\mathfrak{F}^l_{k;i},$$
$$\mathscr{B}^a_{k;j\|i}=\frac{\partial}{\partial \eta^i}(\mathscr{B}^a_{k;j})+\mathscr{B}^b_{k;j}\mathscr{F}^a_{bi}-\mathscr{B}^a_{k;l}\mathfrak{F}^l_{ji}-\mathscr{B}^a_{l;j}\mathfrak{F}^l_{ki},$$
$$\mathscr{B}^a_{ki|j}=\frac{\delta}{\delta w^j}(\mathscr{B}^a_{ki})+\mathscr{B}^b_{ki}\mathscr{F}^a_{b;j}-\mathscr{B}^a_{kl}\mathfrak{F}^l_{i;j}-\mathscr{B}^a_{li}\mathfrak{F}^l_{k;j},$$
$$\mathscr{B}^a_{ki\|j}=\frac{\partial}{\partial \eta^j}(\mathscr{B}^a_{ki})+\mathscr{B}^b_{ki}\mathscr{F}^a_{bj}-\mathscr{B}^a_{kl}\mathfrak{F}^l_{ij}-\mathscr{B}^a_{li}\mathfrak{F}^l_{kj}$$

the $h(v)$-relative covariant derivatives of $\mathscr{B}^a_{k;j}$ and $\mathscr{B}^a_{ki}$ with respect to the connection pair (*IFC*, *NFC*), respectively.

Since the induced Finsler connection *IFC* coincides with the Chern–Finsler connection *CFC* on $(\mathcal{M},\mathcal{F})$, the (2, 0)-curvature of *IFC* vanishes identically, i.e.,

$$R\left(\frac{\delta}{\delta w^i},\frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k}=R\left(\frac{\partial}{\partial \eta^i},\frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k}=R\left(\frac{\partial}{\partial \eta^i},\frac{\partial}{\partial \eta^j}\right)\frac{\partial}{\partial \eta^k}=0.$$

Using (1.8), (3.5) and the vanishing of the (2,0)-curvature of the Chern–Finsler connection $CFC$ on $(M, F)$, we have

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k} = R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta w^j}\right)\frac{\partial}{\partial \eta^k} = R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \eta^j}\right)\frac{\partial}{\partial \eta^k} = 0.$$

Thus we obtain

$$(\mathscr{B}^a_{k;i}\mathscr{A}^l_{a;j} - \mathscr{B}^a_{k;j}\mathscr{A}^l_{a;i})\frac{\partial}{\partial \eta^l} + (\mathscr{B}^a_{k;j|i} - \mathscr{B}^a_{k;i|j} + \mathcal{T}^l_{;ji}\mathscr{B}^a_{k;l})B_a = 0, \tag{4.16}$$

$$(\mathscr{B}^a_{ki}\mathscr{A}^l_{a;j} - \mathscr{B}^a_{k;j}\mathscr{A}^l_{ai})\frac{\partial}{\partial \eta^l} + (\mathscr{B}^a_{k;j\|i} - \mathscr{B}^a_{ki|j} + \mathcal{T}^l_{i;j}\mathscr{B}^a_{k;l})B_a = 0, \tag{4.17}$$

$$(\mathscr{B}^a_{ki}\mathscr{A}^l_{aj} - \mathscr{B}^a_{kj}\mathscr{A}^l_{ai})\frac{\partial}{\partial \eta^l} + (\mathscr{B}^a_{kj\|i} - \mathscr{B}^a_{ki\|j})B_a = 0. \tag{4.18}$$

**Proposition 4.2.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold endowed with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be the complex Finsler submanifold of $(M, F)$ endowed with the induced Finsler connection $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$. Then we have the following identities*

$$\mathscr{B}^a_{k;i}\mathscr{A}^l_{a;j} - \mathscr{B}^a_{k;j}\mathscr{A}^l_{a;i} = 0, \tag{4.19}$$

$$\mathscr{B}^a_{ki}\mathscr{A}^l_{a;j} - \mathscr{B}^a_{k;j}\mathscr{A}^l_{ai} = 0, \tag{4.20}$$

$$\mathscr{B}^a_{ki}\mathscr{A}^l_{aj} - \mathscr{B}^a_{kj}\mathscr{A}^l_{ai} = 0, \tag{4.21}$$

$$\mathscr{B}^a_{k;j|i} - \mathscr{B}^a_{k;i|j} + \mathcal{T}^l_{;ji}\mathscr{B}^a_{k;l} = 0, \tag{4.22}$$

$$\mathscr{B}^a_{k;j\|i} - \mathscr{B}^a_{ki|j} + \mathcal{T}^l_{i;j}\mathscr{B}^a_{k;l}, \tag{4.23}$$

$$\mathscr{B}^a_{kj\|i} - \mathscr{B}^a_{ki\|j} = 0. \tag{4.24}$$

**Proof.** Since $\{\frac{\partial}{\partial \eta^l}, B_a\}$ are the local frames of $\mathcal{V}(\tilde{M}^*)$ with respect to the direct sum decomposition

$$\mathcal{V}(\tilde{M}^*) = \mathcal{V}(\tilde{\mathcal{M}}) \oplus \mathcal{V}(\tilde{\mathcal{M}})^\perp.$$

It follows from (4.16)–(4.18) that we have (4.19)–(4.24). ■

Now using Gauss formula (3.17), (3.18) and Weingarten formula (3.26), (3.28), we have

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial \eta^k} = (R^l_{k;i\bar{j}} + \mathscr{B}^a_{k;i}\mathscr{A}^l_{a;\bar{j}})\frac{\partial}{\partial \eta^l} - (\mathscr{B}^a_{k;i|\bar{j}} - \dot{\mathcal{T}}^l_{;i\bar{j}}\mathscr{B}^a_{kl})B_a, \tag{4.25}$$

$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial \eta^k} = (R^l_{ki;\bar{j}} + \mathscr{B}^a_{ki}\mathscr{A}^l_{a;\bar{j}})\frac{\partial}{\partial \eta^l} - \mathscr{B}^a_{ki|\bar{j}}B_a, \tag{4.26}$$

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial \eta^k} = (R^l_{k\bar{j};i} + \mathscr{B}^a_{k;i}\mathscr{A}^l_{a\bar{j}})\frac{\partial}{\partial \eta^l} - (\mathscr{B}^a_{k;i\|\bar{j}} - \dot{\mathcal{T}}^l_{\bar{j};i}\mathscr{B}^a_{kl})B_a, \tag{4.27}$$

$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial \eta^k} = (R^l_{ki\bar{j}} + \mathscr{B}^a_{ki}\mathscr{A}^l_{a\bar{j}})\frac{\partial}{\partial \eta^l} - \mathscr{B}^a_{ki\|\bar{j}}B_a, \tag{4.28}$$

where we have denoted by

$$\mathscr{B}^a_{k;i|\bar{j}} = \frac{\delta}{\delta \bar{w}^j}(\mathscr{B}^a_{k;i}) + \mathscr{B}^b_{k;i}\mathscr{F}^a_{b;\bar{j}},$$

$$\mathscr{B}^a_{k;i\|\bar{j}} = \frac{\partial}{\partial \bar{\eta}^j}(\mathscr{B}^a_{k;i}) + \mathscr{B}^b_{k;i}\mathscr{F}^a_{b\bar{j}},$$

$$\mathscr{B}^a_{ki|\bar{j}} = \frac{\delta}{\delta \bar{w}^j}(\mathscr{B}^a_{ki}) + \mathscr{B}^b_{ki}\mathscr{F}^a_{b;\bar{j}},$$

$$\mathscr{B}^a_{ki\|\bar{j}} = \frac{\partial}{\partial \bar{\eta}^j}(\mathscr{B}^a_{ki}) + \mathscr{B}^b_{ki}\mathscr{F}^a_{b\bar{j}}$$

the $h(v)$-relative covariant derivatives of $\mathscr{B}^a_{k;i}$ and $\mathscr{B}^a_{ki}$ with respect to the connection pair ($IFC$, $NFC$), respectively. On the other hand, substitute (1.8) in (4.1) and (4.2), we have

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial \eta^k} = B^\gamma_k R^{*\sigma}_{\gamma;i\bar{j}}\frac{\partial}{\partial v^\sigma}, \qquad R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial \eta^k} = B^\gamma_k R^{*\sigma}_{\gamma i;\bar{j}}\frac{\partial}{\partial v^\sigma}, \tag{4.29}$$

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial \eta^k} = B^\gamma_k R^{*\sigma}_{\gamma\bar{j};i}\frac{\partial}{\partial v^\sigma}, \qquad R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial \eta^k} = B^\gamma_k R^{*\sigma}_{\gamma i\bar{j}}\frac{\partial}{\partial v^\sigma}. \tag{4.30}$$

Substitute (3.4) into (4.29) and (4.30) and comparing the obtained results with (4.25)–(4.28), we have the following theorem.

**Theorem 4.3.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold endowed with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be the complex Finsler submanifold of $(M, F)$ endowed with the induced Finsler connection $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$. Then we have the following assertions:*

(1) *The local expression of the Gauss equations are given by*

$$B^\gamma_k \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma;i\bar{j}} = R^l_{k;i\bar{j}} + \mathscr{B}^a_{k;i}\mathscr{A}^l_{a;\bar{j}}, \tag{4.31}$$

$$B^\gamma_k \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma i;\bar{j}} = R^l_{ki;\bar{j}} + \mathscr{B}^a_{ki}\mathscr{A}^l_{a;\bar{j}}, \tag{4.32}$$

$$B^\gamma_k \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma\bar{j};i} = R^l_{k\bar{j};i} + \mathscr{B}^a_{k;i}\mathscr{A}^l_{a\bar{j}}, \tag{4.33}$$

$$B^\gamma_k \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma i\bar{j}} = R^l_{ki\bar{j}} + \mathscr{B}^a_{ki}\mathscr{A}^l_{a\bar{j}}, \tag{4.34}$$

(2) *The local expression of the $\mathscr{B}$-Codazzi equations are given by*

$$B^\gamma_k \mathcal{B}^a_\sigma R^{*\sigma}_{\gamma;i\bar{j}} = -(\mathscr{B}^a_{k;i|\bar{j}} - \dot{\mathcal{T}}^l_{;i\bar{j}}\mathscr{B}^a_{kl}), \tag{4.35}$$

$$B^\gamma_k \mathcal{B}^a_\sigma R^{*\sigma}_{\gamma i;\bar{j}} = -\mathscr{B}^a_{ki|\bar{j}}, \tag{4.36}$$

$$B^\gamma_k \mathcal{B}^a_\sigma R^{*\sigma}_{\gamma\bar{j};i} = -(\mathscr{B}^a_{k;i\|\bar{j}} - \dot{\mathcal{T}}^l_{\bar{j};i}\mathscr{B}^a_{kl}), \tag{4.37}$$

$$B^\gamma_k \mathcal{B}^a_\sigma R^{*\sigma}_{\gamma i\bar{j}} = -\mathscr{B}^a_{ki\|\bar{j}}. \tag{4.38}$$

Next we shall derive the $\mathscr{A}$-Codazzi equations and Ricci equations locally. We first deal with the $(2,0)$-curvature forms of the normal Finsler connection $NFC$. Although the $(2,0)$-curvature forms of $CFC$ vanishes identically, the $(2,0)$-curvature forms of $NFC$ not necessary vanishes.

Let us denote by $R^\perp$ the curvature forms of $NFC$. Locally, if we put

$$R^\perp\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta w^j}\right)B_a = R^{\perp b}_{a;ji}B_b,$$

$$R^\perp\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta w^j}\right)B_a = R^{\perp b}_{ai;j}B_b,$$

$$R^\perp\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \eta^j}\right)B_a = R^{\perp b}_{aji}B_b.$$

Then using Gauss formula (3.17), (3.18) and Weingarten formula (3.25) and (3.27), we deduce that

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta w^j}\right)B_a = (R^{\perp b}_{a;ji} - \mathscr{A}^l_{a;j}\mathscr{B}^b_{l;i} + \mathscr{A}^l_{a;i}\mathscr{B}^b_{l;j})B_b + (\mathscr{A}^l_{a;i|j} - \mathscr{A}^l_{a;j|i} - \mathcal{T}^k_{;ji}\mathscr{A}^l_{a;k})\frac{\partial}{\partial \eta^l}, \tag{4.39}$$

$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta w^j}\right)B_a = (R^{\perp b}_{ai;j} - \mathscr{A}^l_{a;j}\mathscr{B}^b_{li} + \mathscr{A}^l_{ai}\mathscr{B}^b_{l;j})B_b + (\mathscr{A}^l_{ai|j} - \mathscr{A}^l_{a;j\|i} - \mathcal{T}^k_{i;j}\mathscr{A}^l_{a;k})\frac{\partial}{\partial \eta^l}, \tag{4.40}$$

$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \eta^j}\right)B_a = (R^{\perp b}_{aji} - \mathscr{A}^l_{aj}\mathscr{B}^b_{li} + \mathscr{A}^l_{ai}\mathscr{B}^b_{lj})B_b + (\mathscr{A}^l_{ai\|j} - \mathscr{A}^l_{aj\|i})\frac{\partial}{\partial \eta^l}, \tag{4.41}$$

where we have denoted by

$$\mathscr{A}^l_{a;j|i} = \frac{\delta}{\delta w^i}(\mathscr{A}^l_{a;j}) + \mathscr{A}^k_{a;j}\mathfrak{F}^l_{k;i} - \mathscr{A}^l_{b;j}\mathscr{F}^b_{a;i} - \mathscr{A}^l_{a;k}\mathfrak{F}^k_{j;i},$$
$$\mathscr{A}^l_{a;j\|i} = \frac{\partial}{\partial \eta^i}(\mathscr{A}^l_{a;j}) + \mathscr{A}^k_{a;j}\mathfrak{F}^l_{ki} - \mathscr{A}^l_{b;j}\mathscr{F}^b_{ai} - \mathscr{A}^l_{a;k}\mathfrak{F}^k_{ji},$$
$$\mathscr{A}^l_{aj|i} = \frac{\delta}{\delta w^i}(\mathscr{A}^l_{aj}) + \mathscr{A}^k_{aj}\mathfrak{F}^l_{k;i} - \mathscr{A}^l_{bj}\mathscr{F}^b_{a;i} - \mathscr{A}^l_{ak}\mathfrak{F}^k_{j;i},$$
$$\mathscr{A}^l_{aj\|i} = \frac{\partial}{\partial \eta^i}(\mathscr{A}^l_{aj}) + \mathscr{A}^k_{aj}\mathfrak{F}^l_{k;i} - \mathscr{A}^l_{bj}\mathscr{F}^b_{ai} - \mathscr{A}^l_{ak}\mathfrak{F}^k_{ji}$$

the $h(v)$-relative covariant derivatives of $\mathscr{A}^l_{a;j}$ and $\mathscr{A}^l_{aj}$ with respect to the connection pair ($IFC$, $NFC$), respectively.

Now we consider the non-zero (1, 1)-curvature forms of $NFC$. Locally, if we put

$$R^\perp\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)B_a = R^{\perp b}_{a;i\bar{j}}B_b,$$
$$R^\perp\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right)B_a = R^{\perp b}_{ai;\bar{j}}B_b,$$
$$R^\perp\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)B_a = R^{\perp b}_{a\bar{j};i}B_b,$$
$$R^\perp\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)B_a = R^{\perp b}_{ai\bar{j}}B_b.$$

Then using Gauss formula (3.17), (3.18) and Weingarten formula (3.26), (3.28), we obtain

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)B_a = (R^{\perp b}_{a;i\bar{j}} - \mathscr{A}^l_{a;\bar{j}}\mathscr{B}^b_{l;i})B_b + \left[\mathscr{A}^l_{a;i|\bar{j}} - \mathscr{A}^l_{a;\bar{j}|i} - (\dot{\mathcal{T}}^k_{;i\bar{j}}\mathscr{A}^l_{ak} + \dot{\mathcal{T}}^{\bar{k}}_{;i\bar{j}}\mathscr{A}^l_{a\bar{k}})\right]\frac{\partial}{\partial \eta^l}, \tag{4.42}$$
$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right)B_a = (R^{\perp b}_{ai;\bar{j}} - \mathscr{A}^l_{a;\bar{j}}\mathscr{B}^b_{li})B_b + (\mathscr{A}^l_{ai|\bar{j}} - \mathscr{A}^l_{a;\bar{j}\|i} + \dot{\mathcal{T}}^{\bar{k}}_{i;\bar{j}}\mathscr{A}^l_{a\bar{k}})\frac{\partial}{\partial \eta^l}, \tag{4.43}$$
$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)B_a = (R^{\perp b}_{a\bar{j};i} - \mathscr{A}^l_{a\bar{j}}\mathscr{B}^b_{l;i})B_b + (\mathscr{A}^l_{ai\|\bar{j}} - \mathscr{A}^l_{a\bar{j}|i} - \dot{\mathcal{T}}^k_{\bar{j};i}\mathscr{A}^l_{ak})\frac{\partial}{\partial \eta^l}, \tag{4.44}$$
$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)B_a = (R^{\perp b}_{ai\bar{j}} - \mathscr{A}^l_{a\bar{j}}\mathscr{B}^b_{li})B_b + (\mathscr{A}^l_{ai\|\bar{j}} - \mathscr{A}^l_{a\bar{j}\|i})\frac{\partial}{\partial \eta^l}, \tag{4.45}$$

where we have denoted by

$$\mathscr{A}^l_{a;i|\bar{j}} = \frac{\delta}{\delta \bar{w}^j}(\mathscr{A}^k_{a;i}) - \mathscr{F}^b_{a;\bar{j}}\mathscr{A}^l_{b;i},$$
$$\mathscr{A}^l_{a;\bar{j}|i} = \frac{\delta}{\delta w^i}(\mathscr{A}^k_{a;\bar{j}}) + \mathscr{A}^k_{a;\bar{j}}\mathfrak{F}^l_{k;i} - \mathscr{A}^l_{b;\bar{j}}\mathscr{F}^b_{a;i},$$
$$\mathscr{A}^l_{a;\bar{j}\|i} = \frac{\partial}{\partial \eta^i}(\mathscr{A}^l_{a;\bar{j}}) + \mathscr{A}^k_{a;\bar{j}}\mathfrak{F}^l_{ki} - \mathscr{A}^l_{b;\bar{j}}\mathscr{F}^b_{ai},$$
$$\mathscr{A}^l_{a\bar{j}|i} = \frac{\delta}{\delta w^i}(\mathscr{A}^l_{a\bar{j}}) + \mathscr{A}^k_{a\bar{j}}\mathfrak{F}^l_{k;i} - \mathscr{A}^l_{b;\bar{j}}\mathscr{F}^b_{a;i},$$
$$\mathscr{A}^l_{a\bar{j}\|i} = \frac{\partial}{\partial \eta^i}(\mathscr{A}^l_{a\bar{j}}) + \mathscr{A}^k_{a\bar{j}}\mathfrak{F}^l_{ki} - \mathscr{A}^l_{b\bar{j}}\mathscr{F}^b_{ai},$$
$$\mathscr{A}^l_{ai|\bar{j}} = \frac{\delta}{\delta \bar{w}^j}(\mathscr{A}^l_{ai}) - \mathscr{A}^l_{bi}\mathcal{F}^b_{a;\bar{j}},$$
$$\mathscr{A}^l_{ai\|\bar{j}} = \frac{\partial}{\partial \bar{\eta}^j}(\mathscr{A}^l_{ai}) - \mathscr{A}^l_{bi}\mathscr{F}^b_{a\bar{j}}$$

the $h(v)$-relative covariant derivatives of $\mathscr{A}^l_{a;i}$, $\mathscr{A}^l_{a;\bar{j}}$, $\mathscr{A}^l_{a\bar{j}}$, $\mathscr{A}^l_{ai}$ with respect to the connection pair $(IFC, NFC)$, respectively.

**Theorem 4.4.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold endowed with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be the complex Finsler submanifold of $(M, F)$ endowed with the induced Finsler connection $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$. Then we have the following assertions:*

(1) *The local expression of the $\mathscr{A}$-Codazzi equations are given by*

$$\mathscr{A}^l_{a;i|j} = \mathscr{A}^l_{a;j|i} + \mathcal{T}^k_{;ji}\mathscr{A}^l_{a;k}, \tag{4.46}$$

$$\mathscr{A}^l_{ai|j} = \mathscr{A}^l_{a;j\|i} + \mathcal{T}^k_{i;j}\mathscr{A}^l_{a;k}, \tag{4.47}$$

$$\mathscr{A}^l_{ai\|j} = \mathscr{A}^l_{aj\|i}, \tag{4.48}$$

$$B^\gamma_a \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma;i\bar{j}} = \mathscr{A}^l_{a;i|\bar{j}} - \mathscr{A}^l_{a;\bar{j}|i} - (\dot{\mathcal{T}}^k_{;i\bar{j}}\mathscr{A}^l_{ak} + \dot{\mathcal{T}}^{\bar{k}}_{;i\bar{j}}\mathscr{A}^l_{a\bar{k}}), \tag{4.49}$$

$$B^\gamma_a \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma i;\bar{j}} = \mathscr{A}^l_{ai|\bar{j}} - \mathscr{A}^l_{a;\bar{j}\|i} + \dot{\mathcal{T}}^{\bar{k}}_{i;\bar{j}}\mathscr{A}^l_{a\bar{k}}, \tag{4.50}$$

$$B^\gamma_a \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma\bar{j};i} = \mathscr{A}^l_{ai\|\bar{j}} - \mathscr{A}^l_{a\bar{j}|i} - \dot{\mathcal{T}}^k_{\bar{j};i}\mathscr{A}^l_{ak}, \tag{4.51}$$

$$B^\gamma_a \mathcal{B}^l_\sigma R^{*\sigma}_{\gamma i\bar{j}} = \mathscr{A}^l_{ai\|\bar{j}} - \mathscr{A}^l_{a\bar{j}\|i}. \tag{4.52}$$

(2) *The local expression of the Ricci equations are given by*

$$R^{\perp b}_{a;ji} = \mathscr{A}^l_{a;j}\mathscr{B}^b_{l;i} - \mathscr{A}^l_{a;i}\mathscr{B}^b_{l;j}, \tag{4.53}$$

$$R^{\perp b}_{ai;j} = \mathscr{A}^l_{a;j}\mathscr{B}^b_{li} - \mathscr{A}^l_{ai}\mathscr{B}^b_{l;j}, \tag{4.54}$$

$$R^{\perp b}_{aji} = \mathscr{A}^l_{aj}\mathscr{B}^b_{li} - \mathscr{A}^l_{ai}\mathscr{B}^b_{lj}, \tag{4.55}$$

$$B^\gamma_a \mathcal{B}^b_\sigma R^{*\sigma}_{\gamma;i\bar{j}} = R^{\perp b}_{a;i\bar{j}} - \mathscr{A}^l_{a;\bar{j}}\mathscr{B}^b_{l;i}, \tag{4.56}$$

$$B^\gamma_a \mathcal{B}^b_\sigma R^{*\sigma}_{\gamma i;\bar{j}} = R^{\perp b}_{ai;\bar{j}} - \mathscr{A}^l_{a;\bar{j}}\mathscr{B}^b_{li}, \tag{4.57}$$

$$B^\gamma_a \mathcal{B}^b_\sigma R^{*\sigma}_{\gamma\bar{j};i} = R^{\perp b}_{a\bar{j};i} - \mathscr{A}^l_{a\bar{j}}\mathscr{B}^b_{l;i}, \tag{4.58}$$

$$B^\gamma_a \mathcal{B}^b_\sigma R^{*\sigma}_{\gamma i\bar{j}} = R^{\perp b}_{ai\bar{j}} - \mathscr{A}^l_{a\bar{j}}\mathscr{B}^b_{li}. \tag{4.59}$$

**Proof.** First, using (1.8) and (3.5), $B_a = B^\gamma_a \frac{\partial}{\partial v^\gamma}$ and the vanishing of the $(2, 0)$-curvature forms of $CFC$, we have

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta w^j}\right) B_a = R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta w^j}\right) B_a = R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \eta^j}\right) B_a = 0. \tag{4.60}$$

Notice that $\{\frac{\partial}{\partial \eta^l}, B_a\}$ is the local frame of the vector bundle $\mathcal{V}(\tilde{M}^*) = \mathcal{V}(\tilde{\mathcal{M}}) \oplus \mathcal{V}(\tilde{\mathcal{M}})^\perp$. Thus (4.60) together with (4.39)–(4.41) implies (4.46)–(4.48) and (4.53)–(4.55).

On the other hand, we have

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right) B_a = B^\gamma_a R^*\left(\frac{\delta}{\delta w^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial v^\gamma} = B^\gamma_a R^{*\sigma}_{\gamma;i\bar{j}}\frac{\partial}{\partial v^\sigma}, \tag{4.61}$$

$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right) B_a = B^\gamma_a R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\delta}{\delta \bar{w}^j}\right)\frac{\partial}{\partial v^\gamma} = B^\gamma_a R^{*\sigma}_{\gamma i;\bar{j}}\frac{\partial}{\partial v^\sigma}, \tag{4.62}$$

$$R^*\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right) B_a = B^\gamma_a R^*\left(\frac{\delta}{\delta w^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial v^\gamma} = B^\gamma_a R^{*\sigma}_{\gamma\bar{j};i}\frac{\partial}{\partial v^\sigma}, \tag{4.63}$$

$$R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right) B_a = B^\gamma_a R^*\left(\frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \bar{\eta}^j}\right)\frac{\partial}{\partial v^\gamma} = B^\gamma_a R^{*\sigma}_{\gamma i\bar{j}}\frac{\partial}{\partial v^\sigma}. \tag{4.64}$$

Thus (4.49)–(4.52) and (4.56)–(4.59) follow by substituting (3.4) in (4.61)–(4.64) and taking corresponding components in $\mathcal{V}(\tilde{M}^*) = \mathcal{V}(\tilde{\mathcal{M}}) \oplus \mathcal{V}(\tilde{\mathcal{M}})^{\perp}$, respectively. ■

## 5. Torsion of the induced Finsler connection

In this section we obtain the structure equations of the torsion $\mathcal{T}$ of the complex linear connection that associated to *IFC*. Note that the non-zero components of the torsion $\mathcal{T}$ are given by (4.14) and (4.15).

Using (1.8) and (3.5) and (2.14)–(2.17) we have

$$\begin{aligned} T\left(\frac{\delta}{\delta w^k}, \frac{\delta}{\delta w^j}\right) &= T\left(B_k^\alpha \frac{\delta}{\delta z^\alpha} + H_k^a B_a, B_j^\beta \frac{\delta}{\delta z^\beta} + H_j^b B_b\right) \\ &= B_k^\alpha B_j^\beta T\left(\frac{\delta}{\delta z^\alpha}, \frac{\delta}{\delta z^\beta}\right) + B_k^\alpha H_j^a B_a^\beta T\left(\frac{\delta}{\delta z^\alpha}, \frac{\partial}{\partial v^\beta}\right) + B_j^\alpha H_k^a B_a^\beta T\left(\frac{\partial}{\partial v^\beta}, \frac{\delta}{\delta z^\alpha}\right) \\ &= [B_k^\alpha B_j^\beta T^\gamma_{;\beta\alpha} + (B_j^\alpha H_k^a - B_k^\alpha H_j^a) B_a^\beta T^\gamma_{\beta;\alpha}] \frac{\delta}{\delta z^\gamma}, \end{aligned} \tag{5.1}$$

$$T\left(\frac{\partial}{\partial \eta^k}, \frac{\delta}{\delta w^j}\right) = B_k^\alpha B_j^\beta T^\gamma_{\alpha;\beta} \frac{\delta}{\delta z^\gamma}, \tag{5.2}$$

$$T\left(\frac{\delta}{\delta w^k}, \frac{\delta}{\delta \bar{w}^j}\right) = \left(B_k^\alpha B_{\bar{j}}^{\bar{\beta}} \dot{T}^\gamma_{;\alpha\bar{\beta}} + B_k^\alpha H_{\bar{j}}^{\bar{a}} B_{\bar{a}}^{\bar{\beta}} \dot{T}^\gamma_{\bar{\beta};\alpha}\right) \frac{\partial}{\partial v^\gamma} + \left(B_k^\alpha B_{\bar{j}}^{\bar{\beta}} \dot{T}^{\bar{\gamma}}_{;\alpha\bar{\beta}} - B_k^\alpha B_{\bar{j}}^{\bar{\beta}} H_k^a \dot{T}^{\bar{\gamma}}_{\alpha;\bar{\beta}}\right) \frac{\partial}{\partial \bar{v}^\gamma}, \tag{5.3}$$

$$T\left(\frac{\delta}{\delta w^k}, \frac{\partial}{\partial \bar{\eta}^j}\right) = B_k^\alpha B_{\bar{j}}^{\bar{\beta}} T^\gamma_{\bar{\beta};\alpha} \frac{\partial}{\partial v^\gamma}. \tag{5.4}$$

On the other hand, using (2.13) we have

$$\begin{aligned} T\left(\frac{\delta}{\delta w^k}, \frac{\delta}{\delta w^j}\right) &= D_{\frac{\delta}{\delta w^k}} \tilde{v}\left(\frac{\delta}{\delta w^j}\right) - D_{\frac{\delta}{\delta w^j}} \tilde{v}\left(\frac{\delta}{\delta w^k}\right) - \tilde{v}\left[\frac{\delta}{\delta w^k}, \frac{\delta}{\delta w^j}\right] \\ &\quad + \tilde{Q}\left(D_{\frac{\delta}{\delta w^k}} \tilde{Q}\tilde{h}\left(\frac{\delta}{\delta w^j}\right) - D_{\frac{\delta}{\delta w^j}} \tilde{Q}\tilde{h}\left(\frac{\delta}{\delta w^k}\right) - \tilde{Q}\tilde{h}\left[\frac{\delta}{\delta w^k}, \frac{\delta}{\delta w^j}\right]\right). \end{aligned} \tag{5.5}$$

Since by (1.8) and (3.5)

$$\tilde{h}\left(\frac{\partial}{\partial \eta^k}\right) = 0, \quad \tilde{v}\left(\frac{\partial}{\partial \eta^k}\right) = \frac{\partial}{\partial \eta^k}, \quad \tilde{v}\left(\frac{\delta}{\delta w^k}\right) = H_k^a B_a, \quad \tilde{Q}\tilde{h}\left(\frac{\delta}{\delta w^k}\right) = Qh\left(\frac{\delta}{\delta w^k}\right),$$

we have

$$T\left(\frac{\delta}{\delta w^k}, \frac{\delta}{\delta w^j}\right) = D_{\frac{\delta}{\delta w^k}}(H_j^a B_a) - D_{\frac{\delta}{\delta w^j}}(H_k^a B_a) + \tilde{Q}\left(D_{\frac{\delta}{\delta w^k}} \frac{\partial}{\partial \eta^j} - D_{\frac{\delta}{\delta w^j}} \frac{\partial}{\partial \eta^k}\right),$$

where we have used the fact that *IFC* coincides with the Chern–Finsler connection *CFC* of $(\mathcal{M}, \mathcal{F})$ and the adapted horizontal frame $\{\frac{\delta}{\delta w^i}\}$ of the Chern–Finsler connection $\mathcal{D}$ on $(\mathcal{M}, \mathcal{F})$ also satisfies Proposition 2.1. In the following calculations we will repeatedly use those identities in Proposition 2.1 for $\{\frac{\delta}{\delta w^i}\}$ without further statement. Now by the Weingarten formula (3.25) we have

$$D_{\frac{\delta}{\delta w^k}}(H_j^a B_a) = \frac{\delta}{\delta w^k}(H_j^a) B_a + H_j^a \left(-\mathscr{A}^i_{a;k} \frac{\partial}{\partial \eta^i} + \mathscr{F}^b_{a;k} B_b\right).$$

Thus

$$\begin{aligned} &D_{\frac{\delta}{\delta w^k}}(H_j^a B_a) - D_{\frac{\delta}{\delta w^j}}(H_k^a B_a) \\ &= \left[\frac{\delta}{\delta w^k}(H_j^a) - \frac{\delta}{\delta w^j}(H_k^a)\right] B_a + (H_j^b \mathcal{F}^a_{b;k} - H_k^b \mathscr{F}^a_{b;j}) B_a + (H_k^a \mathscr{A}^i_{a;j} - H_j^a \mathscr{A}^i_{a;k}) \frac{\partial}{\partial \eta^i} \\ &= \left[(H^a_{j|k} - H^a_{k|j}) B_a^\gamma + H_i^a \mathcal{T}^i_{;jk} B_a^\gamma + (H_k^a \mathscr{A}^i_{a;j} - H_j^a \mathscr{A}^i_{a;k}) B_i^\gamma\right] \frac{\partial}{\partial v^\gamma}. \end{aligned}$$

Next by Gauss formula (3.17) we have

$$\tilde{Q}\left(D_{\frac{\delta}{\delta w^k}}\frac{\partial}{\partial\eta^j}-D_{\frac{\delta}{\delta w^j}}\frac{\partial}{\partial\eta^k}\right)=\tilde{Q}\left[\mathcal{D}_{\frac{\delta}{\delta w^k}}\frac{\partial}{\partial\eta^j}+\mathscr{B}^a_{j;k}B_a-\mathcal{D}_{\frac{\delta}{\delta w^j}}\frac{\partial}{\partial\eta^k}-\mathscr{B}^a_{k;j}B_a\right]$$
$$=\tilde{Q}\left[\mathcal{T}^i_{;jk}\frac{\partial}{\partial\eta^i}+(\mathscr{B}^a_{j;k}-\mathscr{B}^a_{k;j})B_a\right]$$
$$=\left[\mathcal{T}^i_{;jk}B^\gamma_i+(\mathscr{B}^a_{j;k}-\mathscr{B}^a_{k;j})B^\gamma_a\right]\frac{\delta}{\delta z^\gamma}.$$

Therefore (5.5) reduces to

$$T\left(\frac{\delta}{\delta w^k},\frac{\delta}{\delta w^j}\right)=\left[(H^a_{j|k}-H^a_{k|j})B^\gamma_a+H^a_i\mathcal{T}^i_{;jk}B^\gamma_a+(H^a_k\mathscr{A}^i_{a;j}-H^a_j\mathscr{A}^i_{a;k})B^\gamma_i\right]\frac{\partial}{\partial v^\gamma}$$
$$+\left[\mathcal{T}^i_{;jk}B^\gamma_i+(\mathscr{B}^a_{j;k}-\mathscr{B}^a_{k;j})B^\gamma_a\right]\frac{\delta}{\delta z^\gamma}. \tag{5.6}$$

Using (2.13) and by similar calculations we have

$$T\left(\frac{\partial}{\partial\eta^k},\frac{\delta}{\delta w^j}\right)=\left[(H^a_{j\|k}-\mathscr{B}^a_{k;j}+H^a_i\mathcal{T}^i_{k;j})B^\gamma_a-H^b_j\mathscr{A}^i_{bk}B^\gamma_i\right]\frac{\partial}{\partial v^\gamma}+(\mathscr{B}^a_{jk}B^\gamma_a+\mathcal{T}^i_{k;j}B^\gamma_i)\frac{\delta}{\delta z^\gamma}. \tag{5.7}$$

Notice that

$$\tilde{v}\left[\frac{\delta}{\delta w^k},\frac{\delta}{\delta\bar{w}^j}\right]=\frac{\delta}{\delta\bar{w}^j}(\mathcal{N}^i_k)B^\gamma_i\frac{\partial}{\partial v^\gamma}-\frac{\delta}{\delta w^k}(\mathcal{N}^{\bar{i}}_{\bar{j}})B^{\bar{\gamma}}_{\bar{i}}\frac{\partial}{\partial\bar{v}^\gamma}$$
$$=-\dot{\mathcal{T}}^i_{;k\bar{j}}B^\gamma_i\frac{\partial}{\partial v^\gamma}-\dot{\mathcal{T}}^{\bar{i}}_{;k\bar{j}}B^{\bar{\gamma}}_{\bar{i}}\frac{\partial}{\partial\bar{v}^\gamma}$$

and

$$\tilde{Q}\left\{D_{\frac{\delta}{\delta w^k}}\tilde{Q}\tilde{h}\left(\frac{\delta}{\delta\bar{w}^j}\right)-D_{\frac{\delta}{\delta\bar{w}^j}}\tilde{Q}\tilde{h}\left(\frac{\delta}{\delta w^k}\right)-\tilde{Q}\tilde{h}\left[\frac{\delta}{\delta w^k},\frac{\delta}{\delta\bar{w}^j}\right]\right\}=0.$$

Thus by (2.13) and the Weingarten formula (3.26), we have

$$T\left(\frac{\delta}{\delta w^k},\frac{\delta}{\delta\bar{w}^j}\right)=\left\{H^{\bar{a}}_{\bar{j}|k}B^{\bar{\gamma}}_{\bar{a}}-\left[H^{\bar{a}}_{\bar{j}}\mathscr{A}^{\bar{i}}_{\bar{a};k}-\dot{\mathcal{T}}^{\bar{i}}_{;k\bar{j}}\right]B^{\bar{\gamma}}_{\bar{i}}\right\}\frac{\partial}{\partial\bar{v}^\gamma}-\left\{H^a_{k|\bar{j}}B^\gamma_a-\left[H^a_k\mathscr{A}^i_{a;\bar{j}}+\dot{\mathcal{T}}^i_{;k\bar{j}}\right]B^\gamma_i\right\}\frac{\partial}{\partial v^\gamma}, \tag{5.8}$$

where we have denoted by

$$H^{\bar{a}}_{\bar{j}|k}=\overline{H^a_{j|\bar{k}}},\ \mathscr{A}^{\bar{i}}_{\bar{a};k}=\overline{\mathscr{A}^i_{a;\bar{k}}}.$$

Using (2.13), by similar calculations we derive

$$T\left(\frac{\delta}{\delta w^k},\frac{\partial}{\partial\bar{\eta}^j}\right)=-\left\{H^a_{k\|\bar{j}}B^\gamma_a-\left[H^a_k\mathscr{A}^i_{a\bar{j}}+\dot{\mathcal{T}}^i_{\bar{j};k}\right]B^\gamma_i\right\}\frac{\partial}{\partial v^\gamma}. \tag{5.9}$$

Now comparing the corresponding coefficients in (5.1) and (5.6) we obtain

$$(H^a_{j|k}-H^a_{k|j})B^\gamma_a+H^a_i\mathcal{T}^i_{;jk}B^\gamma_a+(H^a_k\mathscr{A}^i_{a;j}-H^a_j\mathscr{A}^i_{a;k})B^\gamma_i=0 \tag{5.10}$$

and

$$\mathcal{T}^i_{;jk}B^\gamma_i+(\mathscr{B}^a_{j;k}-\mathscr{B}^a_{k;j})B^\gamma_a=B^\alpha_kB^\beta_jT^\gamma_{;\beta\alpha}+(B^\alpha_jH^a_k-B^\alpha_kH^a_j)B^\beta_aT^\gamma_{\beta;\alpha}. \tag{5.11}$$

Contracting (5.10) and (5.11) with $\mathcal{B}^i_\gamma$ and $\mathcal{B}^a_\gamma$, respectively, we get

$$H^a_k\mathscr{A}^i_{a;j}=H^a_j\mathscr{A}^i_{a;k},\qquad H^a_{j|k}-H^a_{k|j}=-H^a_i\mathcal{T}^i_{;jk}, \tag{5.12}$$

$$\mathcal{T}^i_{;jk}=\mathcal{B}^i_\gamma\left[B^\alpha_kB^\beta_jT^\gamma_{;\beta\alpha}+(B^\alpha_jH^a_k-B^\alpha_kH^a_j)B^\beta_aT^\gamma_{\beta;\alpha}\right], \tag{5.13}$$

$$\mathscr{B}^a_{k;j} - \mathscr{B}^a_{j;k} = \mathcal{B}^a_\gamma \left[ B^\alpha_k B^\beta_j T^\gamma_{;\beta\alpha} + (B^\alpha_j H^b_k - B^\alpha_k H^b_j) B^\beta_b T^\gamma_{\beta;\alpha} \right]. \tag{5.14}$$

Comparing the corresponding coefficients in (5.7) and (5.2), we get

$$(H^a_{j\|k} - \mathscr{B}^a_{k;j} + H^a_i \mathcal{T}^i_{k;j}) B^\gamma_a - H^b_j \mathscr{A}^i_{bk} B^\gamma_i = 0, \tag{5.15}$$

$$\mathscr{B}^a_{jk} B^\gamma_a + \mathcal{T}^i_{k;j} B^\gamma_i = B^\alpha_k B^\beta_j T^\gamma_{\alpha;\beta}. \tag{5.16}$$

Contracting (5.15) and (5.16) with $\mathcal{B}^a_\gamma$ and $\mathcal{B}^i_\gamma$, respectively, we get

$$H^a_{j\|k} - \mathscr{B}^a_{k;j} + H^a_i \mathfrak{F}^i_{jk} = 0, \qquad H^b_j \mathscr{A}^i_{bk} = 0, \tag{5.17}$$

$$\mathscr{B}^a_{jk} = \mathcal{B}^a_\gamma B^\alpha_k B^\beta_j T^\gamma_{\alpha;\beta}, \qquad \mathcal{T}^i_{k;j} = \mathcal{B}^i_\gamma B^\alpha_k B^\beta_j T^\gamma_{\alpha;\beta}. \tag{5.18}$$

Comparing the corresponding coefficients in (5.3) and (5.8), we obtain

$$B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^{\bar\gamma}_{;\alpha\bar\beta} - B^\alpha_k B^{\bar\beta}_{\bar j} H^a_k \dot T^{\bar\gamma}_{\alpha;\bar\beta} = H^{\bar a}_{\bar j|k} B^{\bar\gamma}_{\bar a} - \left[ H^{\bar a}_{\bar j} \mathscr{A}^{\bar i}_{\bar a;k} - \dot{\mathcal{T}}^{\bar i}_{;k\bar j} \right] B^{\bar\gamma}_{\bar i} \tag{5.19}$$

and

$$B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{;\alpha\bar\beta} + B^\alpha_k H^{\bar a}_{\bar j} B^{\bar\beta}_{\bar a} \dot T^\gamma_{\bar\beta;\alpha} = -\left\{ H^a_{k|\bar j} B^\gamma_a - \left[ H^a_k \mathscr{A}^i_{a;\bar j} + \dot{\mathcal{T}}^i_{;k\bar j} \right] B^\gamma_i \right\}. \tag{5.20}$$

Contracting (5.19) and (5.20) with $\mathcal{B}^{\bar i}_{\bar\gamma}$ and $\mathcal{B}^{\bar a}_{\bar\gamma}$, respectively, we have

$$\mathcal{B}^{\bar a}_{\bar\gamma} (B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^{\bar\gamma}_{;\alpha\bar\beta} - B^\alpha_k B^{\bar\beta}_{\bar j} H^a_k \dot T^{\bar\gamma}_{\alpha;\bar\beta}) = H^{\bar a}_{\bar j|k}, \tag{5.21}$$

$$\mathcal{B}^{\bar i}_{\bar\gamma} (B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^{\bar\gamma}_{;\alpha\bar\beta} - B^\alpha_k B^{\bar\beta}_{\bar j} H^a_k \dot T^{\bar\gamma}_{\alpha;\bar\beta}) = H^{\bar a}_{\bar j} \mathscr{A}^{\bar i}_{\bar a;k} - \dot{\mathcal{T}}^{\bar i}_{;k\bar j}, \tag{5.22}$$

$$-\mathcal{B}^a_\gamma (B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{;\alpha\bar\beta} + B^\alpha_k H^{\bar a}_{\bar j} B^{\bar\beta}_{\bar a} \dot T^\gamma_{\bar\beta;\alpha}) = H^a_{k|\bar j}, \tag{5.23}$$

$$\mathcal{B}^i_\gamma (B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{;\alpha\bar\beta} + B^\alpha_k H^{\bar a}_{\bar j} B^{\bar\beta}_{\bar a} \dot T^\gamma_{\bar\beta;\alpha}) = H^a_k \mathscr{A}^i_{a;\bar j} + \dot{\mathcal{T}}^i_{;k\bar j}. \tag{5.24}$$

Comparing the corresponding coefficients of (5.4) and (5.9), we get

$$-\left\{ H^a_{k\|\bar j} B^\gamma_a - \left[ H^a_k \mathscr{A}^i_{a\bar j} + \dot{\mathcal{T}}^i_{\bar j;k} \right] B^\gamma_i \right\} = B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{\bar\beta;\alpha}. \tag{5.25}$$

Contracting (5.25) with $\mathcal{B}^a_\gamma$ and $\mathcal{B}^i_\gamma$, respectively, we have

$$H^a_{k\|\bar j} = -\mathcal{B}^a_\gamma B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{\bar\beta;\alpha}, \tag{5.26}$$

$$\dot{\mathcal{T}}^i_{\bar j;k} = \mathcal{B}^i_\gamma B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{\bar\beta;\alpha} - H^a_k \mathscr{A}^i_{a\bar j}. \tag{5.27}$$

From the above calculations, we have

**Proposition 5.1.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$ with the induced Finsler connection $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$. Then we have the following identities*

$$H^a_k \mathscr{A}^i_{a;j} = H^a_j \mathscr{A}^i_{a;k}, H^a_{j|k} - H^a_{k|j} = -H^a_i \mathcal{T}^i_{;jk}, \tag{5.28}$$

$$H^a_{j\|k} = \mathscr{B}^a_{k;j} - H^a_i \mathcal{T}^i_{k;j}, H^b_j \mathscr{A}^i_{bk} = 0, \tag{5.29}$$

$$H^a_{k|\bar j} = -\mathcal{B}^a_\gamma (B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{;\alpha\bar\beta} + B^\alpha_k H^{\bar a}_{\bar j} B^{\bar\beta}_{\bar a} \dot T^\gamma_{\bar\beta;\alpha}), \tag{5.30}$$

$$H^a_{k\|\bar j} = -\mathcal{B}^a_\gamma B^\alpha_k B^{\bar\beta}_{\bar j} \dot T^\gamma_{\bar\beta;\alpha}, \tag{5.31}$$

$$\mathscr{B}^a_{jk} = \mathcal{B}^a_\gamma B^\alpha_k B^\beta_j T^\gamma_{\alpha;\beta}, \tag{5.32}$$

$$\mathscr{B}^a_{k;j} - \mathscr{B}^a_{j;k} = \mathcal{B}^a_\gamma \left[ B^\alpha_k B^\beta_j T^\gamma_{;\beta\alpha} + (B^\alpha_j H^b_k - B^\alpha_k H^b_j) B^\beta_b T^\gamma_{\beta;\alpha} \right]. \tag{5.33}$$

**Theorem 5.2.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold with the Chern–Finsler connection $CFC = (N^\alpha_\mu, F^\alpha_{\beta;\mu}, F^\alpha_{\beta\mu})$ and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$ with the induced Finsler connection $IFC = (\mathcal{N}^i_j, \mathfrak{F}^i_{j;k}, \mathfrak{F}^i_{jk})$. Then the non-vanishing components of the torsion $\mathcal{T}$ of the complex linear connection associated to IFC on $(\mathcal{M}, \mathcal{F})$ and the non-vanishing components of the torsion $T$ of the complex linear connection associated to CFC on $(M, F)$ are related by the following formulas:*

$$\mathcal{T}^i_{;jk} = \mathcal{B}^i_\gamma \left[ B^\alpha_k B^\beta_j T^\gamma_{;\beta\alpha} + (B^\alpha_j H^a_k - B^\alpha_k H^a_j) B^\beta_a T^\gamma_{\beta;\alpha} \right], \tag{5.34}$$

$$\mathcal{T}^i_{k;j} = \mathcal{B}^i_\gamma B^\alpha_k B^\beta_j T^\gamma_{\alpha;\beta}, \tag{5.35}$$

$$\dot{\mathcal{T}}^i_{;k\bar{j}} = H^a_k \mathscr{A}^i_{a;\bar{j}} - \mathcal{B}^i_\gamma (B^\alpha_k B^{\bar{\beta}}_{\bar{j}} \dot{T}^\gamma_{;\alpha\bar{\beta}} + B^\alpha_k B^{\bar{a}}_{\bar{j}} B^{\bar{\beta}}_{\bar{a}} \dot{T}^\gamma_{\bar{\beta};\alpha}), \tag{5.36}$$

$$\dot{\mathcal{T}}^{\bar{i}}_{;k\bar{j}} = H^{\bar{a}}_{\bar{j}} \mathscr{A}^{\bar{i}}_{\bar{a};k} - \mathcal{B}^{\bar{i}}_{\bar{\gamma}} (B^\alpha_k B^{\bar{\beta}}_{\bar{j}} \dot{T}^{\bar{\gamma}}_{;\alpha\bar{\beta}} - B^\alpha_k B^{\bar{\beta}}_{\bar{j}} H^a_k \dot{T}^{\bar{\gamma}}_{\alpha;\bar{\beta}}), \tag{5.37}$$

$$\dot{\mathcal{T}}^i_{\bar{j};k} = B^\alpha_k B^{\bar{\beta}}_{\bar{j}} \mathcal{B}^i_\gamma \dot{T}^\gamma_{\bar{\beta};\alpha} - H^a_k \mathscr{A}^i_{a\bar{j}}. \tag{5.38}$$

## 6. Some application of the fundamental formulas

In this section we shall use the fundamental formulas of the complex Finsler submanifolds to prove some results on complex Finsler submanifolds. A first application is a geometric characterization of the holomorphic curvature of the complex Finsler submanifolds.

Let $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$. Then the holomorphic curvature $\mathcal{K}_{\mathcal{F}}(\eta)$ of $\mathcal{F}$ along $\eta \in \tilde{\mathcal{M}}$ and the holomorphic curvature $K_F(f_*\eta)$ along $f_*\eta \in \tilde{M}$ are given respectively by [1, p. 108], in our notations,

$$\mathcal{K}_{\mathcal{F}}(\eta) = \frac{2}{\mathcal{F}^2(\eta)} g\left( R\left( \eta^i \frac{\delta}{\delta w^i}, \bar{\eta}^j \frac{\delta}{\delta \bar{w}^j} \right) \eta^k \frac{\delta}{\delta w^k}, \eta^l \frac{\delta}{\delta w^l} \right)_\eta, \tag{6.1}$$

$$K_F(f_*\eta) = \frac{2}{F^2(f_*\eta)} \tilde{g}\left( \tilde{R}\left( \eta^i \frac{\delta}{\delta w^i}, \bar{\eta}^j \frac{\delta}{\delta \bar{w}^j} \right) \eta^k \frac{\delta}{\delta w^k}, \eta^l \frac{\delta}{\delta w^l} \right)_{f_*\eta}. \tag{6.2}$$

**Theorem 6.1.** *Let $(M, F)$ be a strongly pseudoconvex complex Finsler manifold and $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of $(M, F)$. Then the holomorphic curvature $\mathcal{K}_{\mathcal{F}}(\eta)$ of $\mathcal{F}$ along $\eta \in \tilde{\mathcal{M}}$ does not exceed the holomorphic curvature $K_F(f_*\eta)$ of $F$ along $f_*\eta \in \tilde{M}$ i.e., we have the following inequality*

$$\mathcal{K}_{\mathcal{F}}(\eta) \le K_F(f_*\eta), \tag{6.3}$$

*and the equality holds if and only if the horizontal second fundamental form $B$ of $(\mathcal{M}, \mathcal{F})$ vanishes identically.*

**Proof.** By a direct calculation, we have

$$\mathcal{K}_{\mathcal{F}}(\eta) = \frac{2}{\mathcal{F}^2(\eta)} g_{l\bar{s}} R^l_{k;i\bar{j}} \eta^i \bar{\eta}^j \eta^k \bar{\eta}^s.$$

On the other hand, we have

$$\tilde{Q}\tilde{h}\left( \eta^k \frac{\delta}{\delta w^k} \right) = \tilde{Q}\left( \eta^k B^\gamma_k \frac{\delta}{\delta z^\gamma} \right) = v^\gamma \frac{\partial}{\partial v^\gamma}.$$

Consequently by (2.19) and (4.3) we get

$$\begin{aligned}\tilde{R}\left(\eta^i\frac{\delta}{\delta w^i},\bar{\eta}^j\frac{\delta}{\delta\bar{w}^j}\right)\eta^k\frac{\delta}{\delta w^k} &= \tilde{Q}\left[\tilde{R}\left(\eta^i\frac{\delta}{\delta w^i},\bar{\eta}^j\frac{\delta}{\delta\bar{w}^j}\right)\tilde{Q}\tilde{h}\left(\eta^k\frac{\delta}{\delta w^k}\right)\right]\\ &= \tilde{Q}\left[\tilde{R}\left(\eta^i\frac{\delta}{\delta w^i},\bar{\eta}^j\frac{\delta}{\delta\bar{w}^j}\right)\left(v^\gamma\frac{\partial}{\partial v^\gamma}\right)\right]\\ &= v^\gamma R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\frac{\delta}{\delta z^\sigma}.\end{aligned}$$

Therefore

$$\begin{aligned}K_F(f_*\eta) &= \frac{2}{F^2(f_*\eta)}\tilde{g}\left(v^\gamma R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\frac{\delta}{\delta z^\sigma},\eta^l\frac{\delta}{\delta w^l}\right)\\ &= \frac{2}{F^2(f_*\eta)}\tilde{g}\left(v^\gamma R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\frac{\delta}{\delta z^\sigma},\eta^l B_l^\nu\frac{\delta}{\delta z^\nu}+\eta^l H_l^a B_a\right)\\ &= \frac{2}{F^2(f_*\eta)}v^\gamma\bar{v}^\nu\tilde{g}_{\sigma\bar{\nu}}R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\\ &= \frac{2}{F^2(f_*\eta)}v^\gamma\tilde{g}_\sigma R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j.\end{aligned}$$

Using Proposition 3.6 and (4.31) in Theorem 4.3 we get

$$B_k^\gamma\mathcal{B}_\sigma^l R^{*\sigma}_{\gamma;i\bar{j}} = R^l_{k;i\bar{j}} + g^{\bar{s}l}\mathscr{B}^a_{k;i}\overline{\mathscr{B}^a_{s;j}},$$

from which we have

$$g_{l\bar{s}}B_k^\gamma\mathcal{B}_\sigma^l R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s = g_{l\bar{s}}R^l_{k;i\bar{j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s + \mathscr{B}^a_{k;i}\overline{\mathscr{B}^a_{s;j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s$$

On the other hand, by (1.17) we obtain

$$\begin{aligned}g_{l\bar{s}}B_k^\gamma\mathcal{B}_\sigma^l R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s &= g_{l\bar{s}}B_k^\gamma\tilde{g}_{\sigma\bar{\beta}}B_{\bar{t}}^{\bar{\beta}}g^{\bar{t}l}R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s\\ &= v^\gamma\tilde{g}_{\sigma\bar{\beta}}B_{\bar{s}}^{\bar{\beta}}R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\bar{\eta}^s\\ &= v^\gamma\bar{v}^\beta\tilde{g}_{\sigma\bar{\beta}}R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j\\ &= v^\gamma\tilde{g}_\sigma R^{*\sigma}_{\gamma;i\bar{j}}\eta^i\bar{\eta}^j.\end{aligned}$$

Since $\mathcal{F}(\eta) = F(f_*\eta)$ and $\mathscr{B}^a_{k;i}\overline{\mathscr{B}^a_{s;j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s = (\mathscr{B}^a_{k;i}\eta^k\eta^i)\overline{(\mathscr{B}^a_{k;i}\eta^k\eta^i)} \geq 0$, we have

$$K_F(f_*\eta) = \mathcal{K}_{\mathcal{F}}(\eta) + \frac{2}{\mathcal{F}^2(\eta)}\mathscr{B}^a_{k;i}\overline{\mathscr{B}^a_{s;j}}\eta^i\bar{\eta}^j\eta^k\bar{\eta}^s \geq \mathcal{K}_{\mathcal{F}}(\eta),$$

which completes the proof. ■

As is known a complex submanifold of a Kähler manifold is also a Kähler manifold. However this is not the case in complex Finsler space. According to [1, p. 95], a pseudoconvex complex Finsler manifold $(M, F)$ is called strongly Kähler Finsler manifold if and only if

$$F^\gamma_{\alpha;\beta} = F^\gamma_{\beta;\alpha};$$

called Kähler Finsler manifold if and only if

$$F^\gamma_{\alpha;\beta}v^\alpha = F^\gamma_{\beta;\alpha}v^\alpha;$$

called weakly Kähler Finsler manifold if and only if

$$\tilde{g}_\gamma[F^\gamma_{\alpha;\beta} - F^\gamma_{\beta;\alpha}]v^\alpha = 0.$$

These conditions can also expressed in terms of suitable contraction of the horizontal torsion $T^\gamma_{;\alpha\beta}$ with $v^\alpha$. Using (5.34) in Theorem 5.2, we immediately have

**Theorem 6.2.** *A complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ of a strongly Kähler Finsler manifold $(M, F)$ is a strongly Kähler Finsler submanifold if and only if*

$$\mathcal{B}^i_\gamma (B^\alpha_j H^a_k - B^\alpha_k H^a_j) B^\beta_a T^\gamma_{\beta;\alpha} = 0. \tag{6.4}$$

*A complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ of a Kähler Finsler manifold $(M, F)$ is a Kähler Finsler submanifold if and only if*

$$\mathcal{B}^i_\gamma B^\alpha_k (\eta^j H^a_j B^\beta_a) T^\gamma_{\beta;\alpha} = 0. \tag{6.5}$$

*A complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ of a weakly Kähler Finsler manifold $(M, F)$ is a weakly Kähler Finsler submanifold if and only if*

$$g_i \mathcal{B}^i_\gamma B^\alpha_k (\eta^j H^a_j B^\beta_a) T^\gamma_{\beta;\alpha} = 0. \tag{6.6}$$

Theorem 6.2 has a simple form when one restricts oneself to the complex geodesics of the totally geodesic complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ of $(M, F)$. Note that a complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ of $(M, F)$ is called totally geodesic if any complex geodesic of $(\mathcal{M}, \mathcal{F})$ is also a complex geodesic of $(M, F)$.

As is known [1, p. 101], a complex geodesic $\{z^\alpha(t)\}$ of $(M, F)$ satisfies the following equations

$$\frac{\mathrm{d}^2 z^\alpha(t)}{\mathrm{d}t^2} + N^\alpha_\beta(z(t), v(t)) \frac{\mathrm{d}z^\beta(t)}{\mathrm{d}t} = \tilde{\theta}^{*\alpha}, \alpha = 1, \ldots, n, \tag{6.7}$$

where $v^\alpha(t) = \frac{\mathrm{d}z^\alpha(t)}{\mathrm{d}t}$ and $\tilde{\theta}^{*\alpha} = \tilde{g}^{\bar{\nu}\alpha} \tilde{g}_{\beta\bar{\sigma}} (F^{\bar{\sigma}}_{\bar{\mu};\bar{\nu}} - F^{\bar{\sigma}}_{\bar{\nu};\bar{\mu}}) v^\beta \bar{v}^\mu$. Since $\delta v^\alpha = dv^\alpha + N^\alpha_\beta dz^\beta$ the geodesic equations (6.7) can also be expressed as

$$\frac{\delta v^\alpha}{\mathrm{d}t} = \tilde{\theta}^{*\alpha}, \quad \alpha = 1, \ldots, n.$$

Now let $\{w^i(t)\}$ be a complex geodesic of a complex Finsler submanifold $(\mathcal{M}, \mathcal{F})$ of $(M, F)$, then we have

$$\frac{\delta \eta^i}{\mathrm{d}t} = \theta^{*i}, \quad i = 1, \ldots, m,$$

where $\eta^i(t) = \frac{\mathrm{d}w^i(t)}{\mathrm{d}t}$ and $\theta^{*i} = g^{\bar{k}i} \tilde{g}_{s\bar{l}} (\mathcal{F}^{\bar{l}}_{\bar{j};\bar{k}} - \mathcal{F}^{\bar{l}}_{\bar{k};\bar{j}}) \eta^s \bar{\eta}^j$. Since $\delta\eta^i = d\eta^i + \mathcal{N}^i_j dw^j$, we have $\delta\eta^i = \mathcal{B}^i_\alpha \delta v^\alpha$. Therefore

$$\theta^{*i} = \mathcal{B}^i_\alpha \tilde{\theta}^{*\alpha},$$

which implies that if $(M, F)$ is weakly Kähler Finsler along its complex geodesic then $(\mathcal{M}, \mathcal{F})$ is also weakly Kähler Finsler along its complex geodesic. By the relation $\delta v^\alpha = B^\alpha_i \delta\eta^i + B^\alpha_a H^a_i dw^i$ we have

$$\tilde{\theta}^{*\alpha} = B^\alpha_i \theta^{*i} + B^\alpha_a H^a_i \eta^i.$$

**Theorem 6.3.** *Let $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of a weakly Kähler Finsler manifold $(M, F)$. Then $(\mathcal{M}, \mathcal{F})$ is totally geodesic if and only if*

$$B^\alpha_a H^a_i \eta^i = 0, \quad \alpha = 1, \ldots, n$$

*holds along any complex geodesic $\{w^i(t)\}$ of $(\mathcal{M}, \mathcal{F})$.* ■

**Corollary 6.4.** *Let $(\mathcal{M}, \mathcal{F})$ be a complex Finsler submanifold of a weakly Kähler Finsler manifold $(M, F)$. Then $(\mathcal{M}, \mathcal{F})$ is totally geodesic if and only if the horizontal components of the second fundamental form satisfy*

$$(\mathscr{B}^a_{k;j} - \mathscr{B}^a_{j;k}) \eta^k = 0$$

*along any complex geodesic $\{w^i(t)\}$ of $(\mathcal{M}, \mathcal{F})$.* ■

**Remark.** If $F = h_{\alpha\bar{\beta}}(z)v^{\alpha}\overline{v^{\beta}}$ comes from a Kähler metric on $M$ and $\mathcal{M}$ is a complex submanifold of $M$. Then $\tilde{g}_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}}(z)$ are independent of the direction $v \in \tilde{M}$ and $T^{\gamma}_{\beta;\alpha} = F^{\gamma}_{\alpha\beta} \equiv 0$. Thus $\mathscr{B}^{a}_{jk} \equiv 0$ and (6.4)–(6.6) hold identically. In this case our results coincide with the classic results in complex submanifolds of Kähler manifolds. Theorems 6.1, 6.2 and Corollary 6.4 show the importance of the horizontal components of the second fundamental form of $B$ in the investigation of the theory of the complex Finsler submanifolds. More applications of the fundamental formulas of the complex Finsler submanifolds will be investigated in a forthcoming paper.

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